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THE ZIQQURATH
OF EXACT SEQUENCES OF n -GROUPOIDS

MAT\02

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Ai miei genitori Lalla e Saverio,
che hanno sempre creduto in me.
A Valentina,
senza di lei questa Tesi non sarebbe mai stata scritta.

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Chapter 1

Introduction

Higher Dimensional Categories are showing relevant implications in Algebraic Topology (classification of homotopy n -types, operads, cobordism), and in Algebraic Geometry (Grothendieck's n -stacks, non-abelian cohomology), not to mention recent applications in Mathematical Physics (TQFT, higher order gauge theory) and Computer Science.

Nevertheless basic algebraic tools, in order to further develop the theory, are far from being established. A step forward towards this direction would be having an essential understanding of the notion of exactness for Higher Dimensional Categories, and of the limits involved in defining this notion.

1.1 Summary

In 1970 Ronald Brown published a paper [Bro70] on an uprising area of mathematical research: the theory of groupoids.

According to the category-theorist, a *groupoid* is a category with all of its morphisms invertible w.r.t. composition [ML98, Hig05]. In fact the notion of groupoid was introduced earlier as a generalization of the notion of *group*, where the binary operation is only partially defined.

In studying connections with algebraic topology and non-abelian cohomology, Brown showed that, given a fibration F of pointed groupoids and its (strict) kernel \mathbb{K}_s :

$$\mathbb{K}_s \xrightarrow{K} \mathbb{B} \xrightarrow{F} \mathbb{C}$$

it was possible to obtain a 6-term exact sequence

$$\pi_1 \mathbb{K}_s \xrightarrow{\pi_1 K} \pi_1 \mathbb{B} \xrightarrow{\pi_1 F} \pi_1 \mathbb{C} \xrightarrow{\delta} \pi_0 \mathbb{K}_s \xrightarrow{\pi_0 K} \pi_0 \mathbb{B} \xrightarrow{\pi_0 F} \pi_0 \mathbb{C} \quad (1.1)$$

of groups and pointed sets, where π_0 is the functor giving set of isomorphism classes of object, and π_1 the group of endomorphisms of the point.

Still, the condition of F being a fibration was motivated by “the analogy [...] with topological situations” [Bro70]: this influenced the kind of limit considered, *i.e.* a (strict) kernel.

However it suffices to consider the homotopy kernel instead of the strict kernel to remove the need of restricting to fibrations, so that construction above still holds for a generic groupoid morphism.

These ideas developed further in [HKK02, DKV04], where the authors generalized Brown’s result to 2-groupoids. The 2-groupoids considered are weakly invertible 2-categories. Duskin, Kieboom and Vitale showed in [HKK02] that, given a morphism of 2-groupoids $G : \mathbb{C} \rightarrow \mathbb{D}$, it is still possible to get a 6-term sequence as in (1.1) of (strict) categorical groups and pointed groupoids, where π_0 is the classifying-functor, and π_1 gives the cat-group of endomorphisms of the point. This sequence is exact in a suitable sense (see [Vit02]). Further, since $\pi_0 \circ \pi_1 = \pi_1 \circ \pi_0$, applying π_1 and π_0 again we get two 6-term exact sequences that can be pasted together in a 9-term exact sequence, where the left-most three terms are abelian groups, the central three are groups and the right-most three are just pointed sets.

The purpose of this thesis is to extend these results to a n -dimensional context.

The setting is the sesqui-category [Str96] of n -groupoids, strict n -functors and lax n -transformations. We consider a notion of n -groupoid equivalent to that of [KV91], *i.e.* a weakly invertible n -category, but our approach is genuinely recursive.

Being more precise, n -categories and n -functors are given by means of the standard enrichment in the category of $(n-1)$ -categories and $(n-1)$ -functors, w.r.t its cartesian bi-closed structure.

Differently, n -transformations considered come in a lax version, being a direct generalization of those of [Bor94] for 2-categories. In fact a notion of strict natural n -transformation is also sketched, but that has shown inadequate in developing the theory for all morphisms, and not just for fibrations.

The lax n -transformations introduced are equivalent to that considered by Crans in [Cra95], yet inductive definition allows to deal with such morphisms directly, without the complications of the theory of pasting schemes [Joh89] (as in [Cra95]).

The n -groupoids we consider are n -categories which are locally $(n-1)$ -groupoids and whose 1-cells are equivalences. In this setting we introduce a straightforward generalization of π_0 (lower “*” stays for pointed):

$$\pi_0^{(n)} : n\mathbf{Gpd}_* \rightarrow (n-1)\mathbf{Gpd}_*$$

Further, standard h -pullbacks [Mat76a] are introduced, together with their 1-dimensional, *i.e.* ordinary, universal property. This allows to deal with

h -kernels, and to define a *loop*-functor

$$\Omega : n\mathbf{Cat} \rightarrow n\mathbf{Cat}, \quad \mathbb{C} \mapsto \begin{array}{ccc} \Omega(\mathbb{C}) & \xrightarrow{\quad} & \mathbb{I} \\ \downarrow & \nearrow & \downarrow [*] \\ \mathbb{I} & \xrightarrow{[*]} & \mathbb{C} \end{array}$$

which gives the functor

$$\pi_1^{(n)} : n\mathbf{Gpd}_* \rightarrow (n-1)\mathbf{Gpd}_*, \quad \text{with } \pi_1^{(n)}(\mathbb{C}) = \pi_0^{(n)}(\Omega(\mathbb{C}))$$

To extend the last to a (contra-variant) sesqui-functor, it is necessary to consider h -pullbacks with their 2-dimensional universal property, this involving 3-morphisms between n -natural transformations, dimension-raising horizontal composition of 2-morphisms and the necessary algebra for those. To this end, an appropriate notion of sesqui²category has been introduced.

Finally, we set up a notion of exactness for a triple (K, φ, F) in $n\mathbf{Gpd}_*$

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \downarrow \varphi & \swarrow & \\ \mathbb{K} & \xrightarrow{K} & \mathbb{B} & \xrightarrow{F} & \mathbb{C} \end{array}$$

and we prove the

MAIN RESULT

For any natural number n ,

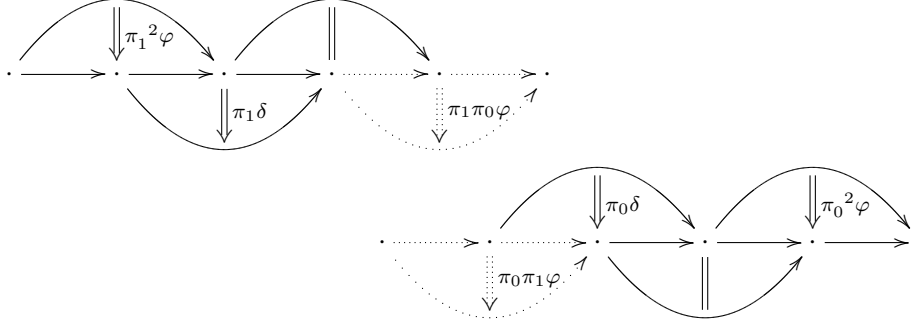
1. Sesqui-functors $\pi_0^{(n)}$ and $\pi_1^{(n)}$ preserve exactness, the last up to reversing the directions of 2-morphisms.
2. Sesqui-functors $\pi_0^{(n)}$ and $\pi_1^{(n)}$ commute, i.e.

$$\pi_1^{(n)} \circ \pi_0^{(n-1)} = \pi_0^{(n)} \circ \pi_1^{(n-1)}.$$

3. Given a morphism of pointed n -groupoids $F : \mathbb{B} \rightarrow \mathbb{C}$, with h -kernel \mathbb{K} , there exists a canonical morphism Δ of pointed $(n-1)$ -groupoids, and 2-morphism δ , such that the sequence below is exact:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & \searrow & \downarrow \delta & \swarrow & \downarrow \pi_0 \varphi & \searrow & \\ \pi_1 \mathbb{K} & \xrightarrow{\pi_1 K} & \pi_1 \mathbb{B} & \xrightarrow{\pi_1 F} & \pi_1 \mathbb{C} & \xrightarrow{\Delta} & \pi_0 \mathbb{K} & \xrightarrow{\pi_0 K} & \pi_0 \mathbb{B} & \xrightarrow{\pi_0 F} & \pi_0 \mathbb{C} \\ & \searrow & \downarrow \pi_1 \varphi & \swarrow & \parallel & \swarrow & \downarrow \pi_0 \varphi & \searrow & \\ & & 0 & & 0 & & & & \end{array}$$

Applying $\pi_0^{(n-1)}$ and $\pi_1^{(n-1)}$, we get two six-term sequences, exact by (1) above. Those can be pasted by (2) above, in a nine-term exact sequence of $(n-2)$ -groupoids (cells to be pasted are dotted in the diagram):



Iterating the process we finally obtain a $9 \cdot n$ exact sequence of pointed sets. Furthermore, since the sesqui-functors π_0 force h -groups structures, the first $9 \cdot (n-1)$ terms are structured as an exact sequence of groups, abelian up to the $9 \cdot (n-2)^{\text{th}}$.

Similarly, the last-but-one step produces a $9 \cdot (n-1)$ term sequence of pointed groupoids, $9 \cdot (n-2)$ of which are indeed categorical groups [Vit02], finally $9 \cdot (n-3)$ are commutative.

Arranging these sequences from top (dimension n) to bottom (dimension 0) we unveil the shape of a *Ziggurath*¹, in which each level is an exact sequence of k -groupoids ($k = 0, \dots, n$). In the relations between contiguous levels are nested classification properties of n -groupoids and their morphisms, many of them are still to be investigated.

1.2 Further developments

The sesqui-categorical setting presented here yields a fruitful perspective in the study of n -dimensional categorical structures. In fact this is a general fact, and it permeates categorical investigations from its very beginning:

“categories are *two* steps away from naturality, the concept they were designed to formalize. [...] From the very study of the established practice of routinely specifying morphisms along with each mathematical structure, we were presented, in the 1940’s, with an extra dimension: morphisms between morphisms. We were naturally led by naturality to objects, arrows *and* 2-cells. [Str96]”

¹Ziqquraths (or Ziggurats) were a type of step pyramid temples common to the inhabitants of ancient Mesopotamia [Sto97, Opp77].

Moreover the inductive approach permits to deal with a sesqui-categorical environment in any dimension, carrying along the constructions while ascending the dimensional ladder.

Two main points are currently being investigated by the author.

First, the native sesqui-categorical setting offers the chance to study weak structures inductively, reducing most of coherence issues to (inductively nested) planar diagrams. This would make possible to describe some lax n -dimensional structures more easily in a pure diagrammatic way, in terms of cells and of compositions given explicitly. Namely, it would be interesting to consider an inductively defined sesqui-categories of n -groupoids with weak units, in order to compare it with the special connected 3-dimensional case of [JK07]. Similarly for other semi-strict versions, as in [Pao07].

Second, an application to abelian chain complexes is announced. In fact, most authors prefer globally defined versions of n -categorical structures, as it makes the internalization process easier. In this way is usually proved the equivalence between ω -categories in **Ab** (the category of abelian groups) and non-negatively graded chain complexes of abelian groups.

Yet it is possible to deal with this equivalence in our context too. In fact, the category of length- n abelian chain complexes is equivalent to the category of abelian group objects in $n\mathbf{Cat}$ ([Lei04], Example 1.4.11). More interestingly this equivalence extends to homotopies and natural n -transformation.

Now, abelian group objects in $n\mathbf{Cat}$ behave well with respect to the h -structure of $n\mathbf{Cat}$, similarly to what is shown for monoid objects in a similar situation by Grandis in [Gra97]. That is the reason why it is worth studying in this new setting how the theory extends to.

1.3 Conventions

The purpose of this section is to make life easier to the reader, stating the notational conventions (and other) adopted. Nevertheless exceptions to these conventions are not rare, although always pointed out.

Doubled capitals as $\mathbb{A}, \mathbb{B}, \mathbb{C}$ are used for n -categories and n -groupoids, capitals as F, G, H for their morphisms, lower case Greek letters α, β, γ for their 2-morphism. Finally capital Greek letters as Σ, Λ are reserved to 3-morphisms.

Objects of a n -category (n -groupoid) are denoted by lower case Latin letters, subscripted by the number “0”. Objects have often the same letter as the

big doubled capital denoting the whole structure, using primes (or other modifiers) for different objects, *e.g.* a_0, a'_0, a''_0 are objects of \mathbb{A} .

Cells follow a similar convention, where the number represent the dimension of the cell, as the 1-cell $b_1 : b_0 \rightarrow b'_0$ of \mathbb{B} . 2-Cells are often represented by double arrows ($b_2 : b_1 \Rightarrow b'_1$). As in higher dimension it would not be quite practical, for representing a k -cell we label the arrow itself with the number k : $b_k : b_{k-1} \multimap b'_{k-1}$.

In order to avoid confusion with the name *morphisms* (reserved for the sorts of the environment sesqui-category) the sorts of our n -categories (n -groupoids) are always named *cells*.

A major exception to these rules is the Chapter on *Sesqui-Categories*. Indeed it uses its own notational conventions, explained therein.

Compositions are dealt with different symbols in order to distinguish the (internal) compositions of cells from the (external) compositions of morphisms.

We adopt for cells-composition the empty circle superscripted by a number that represents the dimension of the intersection cell. Example: $c_h \circ^m c_k$ means that the h -cell c_h and the k -cell c_k are composed along their common boundary, that is a m -cell c_m . In this case we will use the terms m -domain and m -codomain. Let us point out that superscripted m is often omitted, specially when it is 0.

For the compositions of morphisms we use the filled circle superscripted by a number that represents the dimension of the intersection morphisms. In the present work we will use only 0-compositions and 1-compositions of morphism, hence the symbols \bullet^0 and \bullet^1 . They come with a lower-scripted L or R if they are left or right whiskering, respectively. Moreover dimension raising 0-composition of 2-morphisms is denoted $*$.

All composition-symbols are omitted when clear from the context.

The compositions of cells morphisms will be written in algebraic order, *e.g.* for $c_1 : c_0 \rightarrow c'_0$ and $c'_1 : c'_0 \rightarrow c''_0$ we will write $c_1 \circ^0 c'_1 : c_0 \rightarrow c''_0$. The other order is considered as *evaluation*, so parentheses will be used, *e.g.* for $F : \mathbb{A} \rightarrow \mathbb{B}$ and $G : \mathbb{B} \rightarrow \mathbb{C}$ we will write $G(F(-)) : \mathbb{A} \rightarrow \mathbb{C}$.

1.4 Synopsis

The thesis is organized as follows:

the rest of the introduction is dedicated to analyze low-dimensional cases, that inspired this generalization;

the second chapter gives the sesqui-categorical-theoretical framework: different characterizations are compared, finite products and h -pullbacks are introduced with their universal properties;

strict n -categories are defined in the third chapter, together with their morphisms (n -functor) and their 2-morphisms (lax n -transformations); moreover finite products and standard h -pullbacks are constructed explicitly;

the fourth chapter introduces n -groupoids, h -surjective morphisms and equivalences: these are necessary to formulate a notion of exactness for the sesqui-categories of pointed n -groupoids; moreover we extend to them some classical result as the adjunction discrete/iso-classes functor, and one point suspension;

in order to deal with 3-morphisms of n -categories, a new framework is defined in the fifth chapter, namely *sesqui²-categories*; lax n -modifications are introduced thereafter, together with other whiskering and compositions; in fact a dimension raising 0-composition of 2-morphisms is given and many useful algebraic properties are proved;

with the machinery developed in the previous ones, the sixth chapter presents the main result: the construction of (a *Ziqqurath* of) exact sequences in any lower dimension from a given morphism of n -groupoids; this is achieved in few steps, the starting point being a 2-dimensional property of pullbacks that h -pullbacks in $n\mathbf{Cat}$ are proved to satisfy;

finally the appendix contains a comparison with the globular approach, the groupoid condition and the choice of inverses.

1.5 Case study: dimension one

In this section we recall, for reader's convenience, the construction set up in [Bro70].

1.5.1 Browns' result

Let us suppose we are given a functor

$$F : \mathbb{B} \rightarrow \mathbb{C}$$

between two groupoids.

For every fixed object b_0 in \mathbb{B} , F induces a map

$$\mathrm{St}_F(b_0) : \mathrm{St}_{\mathbb{B}}(b_0) \rightarrow \mathrm{St}_{\mathbb{C}}(Fb_0)$$

where $\mathrm{St}_{\mathbb{B}}(b_0) = \bigcup_{b'_0 \in \mathbb{B}_0} \mathbb{B}_1(b_0, b'_0)$.

Suppose now that for every b_0 , the map $\mathrm{St}_F(b_0)$ is surjective. Such F is called *star-surjective*, or *fibration*.

We can consider its (strict) kernel w.r.t. an object b of \mathbb{B} :

$$\mathbb{K}_s \xrightarrow{K} \mathbb{B} \xrightarrow{F} \mathbb{C} \quad (1.2)$$

Here the groupoid \mathbb{K}_s is just the strict fiber over the object Fb of \mathbb{C} , *i.e.* the groupoid with objects b_0 of \mathbb{B} such that $Fb_0 = Fb$, and arrows b_1 of \mathbb{B} such that $Fb_1 = 1_b$. Finally K is the natural inclusion.

Diagram (1.2) above can be *restricted* to automorphisms groups over fixed objects, thus giving the exact sequence

$$1 \longrightarrow \mathbb{K}_s(b, b) \xrightarrow{K^{b,b}} \mathbb{B}(b, b) \xrightarrow{F^{b,b}} \mathbb{C}(Fb, Fb)$$

Proof. Exactness in $\mathbb{K}_s(b, b)$ for K injective, in $\mathbb{B}(b, b)$ for $\mathbb{K}_s = F^{-1}(Fb)$. \square

Furthermore (1.2) gives also an exact sequence of pointed sets, when we apply the *isomorphism-classes-functor*

$$\pi_0 : \mathbf{Gpd} \rightarrow \mathbf{Set}$$

that sends a groupoid in the set of classes of isomorphic objects.

We obtain the diagram

$$\pi_0 \mathbb{K}_s \xrightarrow{\pi_0 K} \pi_0 \mathbb{B} \xrightarrow{\pi_0 F} \pi_0 \mathbb{C}$$

exact in $\pi_0 \mathbb{B}$.

Proof. Clearly $\text{Im}(\pi_0 K) \subseteq \text{Ker}(\pi_0 F)$. Suppose then $\pi_0 F(\{b_0\}) = \{Fb_0\} = \{Fb\}$, where brackets means iso-class. Then the hom-set $\mathbb{C}(Fb, Fb_0)$ is nonempty, containing an element c_1 , say. Star-surjectivity in b_0 implies that there is a an arrow $b_1 : b_0 \rightarrow b'$ such that $F(b_1) = c_1^{-1}$, but this means $\{b_0\} = \{b'\}$. Since $Fb' = Fb$ the proof is complete. \square

Finally we define a morphism of pointed sets

$$\delta : \mathbb{C}(Fb, Fb) \longrightarrow \pi_0 \mathbb{K}_s$$

in the following way: given the arrow $c_1 : Fb \rightarrow Fb$ star-surjectivity yields a $b_1 : b \rightarrow b'$ such that $Fb_1 = c_1$. Then we let $\delta(c_1) = \{b'\}$. Clearly this map is well defined, since for a different lifting of c_1 , its codomain is isomorphic to b' . Moreover it is obviously pointed by the identity.

Now the new sequence connected by δ is everywhere exact

$$1 \longrightarrow \mathbb{K}_s(b, b) \xrightarrow{K^{b,b}} \mathbb{B}(b, b) \xrightarrow{F^{b,b}} \mathbb{C}(Fb, Fb) \xrightarrow{\delta} \pi_0 \mathbb{K}_s \xrightarrow{\pi_0 K} \pi_0 \mathbb{B} \xrightarrow{\pi_0 F} \pi_0 \mathbb{C} \quad (1.3)$$

where the last three terms are pointed sets, the other are groups.

Proof. It remains to prove the exactness in $\mathbb{C}(Fb, Fb)$ and in $\pi_0 \mathbb{K}_s$.

As for the first, let $b_1 : b \rightarrow b$ be given. Then among the liftings of Fb_1 there is b_1 itself, hence $\delta(Fb_1) = \{b\}$. Conversely let $c_1 : Fb \rightarrow Fb$ in the kernel of δ . This means that for a lifting $b_1 : b \rightarrow b'$ of c_1 , $\{b'\} = \{b\}$ in the fiber, *i.e.* there is a $b'_1 : b' \rightarrow b$ such that $F(b'_1) = 1_b$. Then $F(b_1 \circ b'_1) = F(b_1) \circ F(b'_1) = F(b_1) \circ 1_b = F(b_1) = c_1$, with $b_1 \circ b'_1 \in \mathbb{B}(b, b)$. For the second, let a $c_1 : Fb \rightarrow Fb$ be given. Then $\delta(c_1) = \{b'\}$ for a lifting $b_1 : b \rightarrow b'$ of c_1 . Hence $\{b'\} = \{b\}$ in $\pi_0 \mathbb{B}$. Conversely let $\{b_0\}$ in the kernel of $\pi_0 K$. This means $\{b_0\} = \{b\}$ in $\pi_0 \mathbb{B}$, *i.e.* there exists a $b_1 : b \rightarrow b_0$ in \mathbb{B} , which implies $\delta(Fb_1) = \{b_0\}$. \square

A first attempt in extending this to higher dimensional structures has been done by Hardie, Kamps and Kieboom in [HKK02], where they obtain a similar result for fibrations of bigroupoids.

Nevertheless the necessity to consider only fibrations is not just a limitation in the choice of morphisms, but it introduces also serious difficulties in trying to further extend the result to n -groupoids.

On this lines Duskin, Kieboom, and Vitale proposed in [DKV04] the different setting given by considering homotopy kernels instead of kernels. Consequently a new notion of exactness was introduced.

1.5.2 Brown's result revisited

In order to fully understand the generalization of [DKV04], we start by considering the one dimensional construction w.r.t. homotopy kernels instead of strict kernels. Notice that here we provide only constructions since proofs are consequence of our general result.

Let us consider a functor

$$F : \mathbb{B} \rightarrow \mathbb{C}$$

between two groupoids. We define the h -fiber over a chosen object Fb of \mathbb{C}

$$\begin{array}{ccccc} & & [Fb] & & \\ & \swarrow & \Downarrow \varphi & \searrow & \\ \mathbb{K} & \xrightarrow{K} & \mathbb{B} & \xrightarrow{F} & \mathbb{C} \end{array}$$

where

- $[Fb]$ is the constant functor;
- \mathbb{K} is the comma-groupoid with objects the pairs $(b_0, c_1 : Fb \longrightarrow Fb_0)$. An arrow $(b_0, c_1) \rightarrow (b'_0, c'_1)$ is a pair $(b_1, =)$ where $b_1 : b_0 \rightarrow b'_0$, and the “=” stays for the equality $c_1 = Fb_1 \circ c'_1$;
- $K : \mathbb{K} \rightarrow \mathbb{B}$ is the faithful functor defined by

$$K((b_0, c_1)) = b_0, \quad K((b_1, =)) = b_1;$$

- $\varphi : [Fb] \Rightarrow KF$ is the natural isomorphism with components

$$\varphi_{(b_0, c_1)} = c_1 : Fb \rightarrow Fb_0.$$

Again we can get an exact sequence of groups and pointed sets

$$1 \longrightarrow \mathbb{K}(b, b) \xrightarrow{K^{b,b}} \mathbb{B}(b, b) \xrightarrow{F^{b,b}} \mathbb{C}(Fb, Fb) \xrightarrow{\delta} \pi_0 \mathbb{K} \xrightarrow{\pi_0 K} \pi_0 \mathbb{B} \xrightarrow{\pi_0 F} \pi_0 \mathbb{C} \quad (1.4)$$

where the connecting map δ is defined in a natural way

$$\delta(c_1 : Fb \longrightarrow Fb) = (b, c_1)$$

Now Brown's result can be seen as a corollary, and this gives a conceptual insight about the relation between these two different settings.

In fact, there exists a fully faithful functor $I_b : \mathbb{K}_s \rightarrow \mathbb{K}$; this is given explicitly by letting $I_b(b_0) = (b_0, 1_{Fb})$, indeed it is provided by the universal property defining the homotopy kernel as a comparison functor.

Clearly F is a fibration of groupoids if, and only if, for each object b of \mathbb{B} the functor I_b is essentially surjective on objects. Then when F is a fibration, one can replace $\mathbb{K}(b, b)$ and $\pi_0 \mathbb{K}$ by $\mathbb{K}_s(b, b)$ and $\pi_0 \mathbb{K}_s$, and obtain Brown's exact sequence (1.3) from (1.4).

1.6 Case study: dimension two

In [DKV04] the authors prove a similar result for morphisms of 2-groupoids, *i.e.* weakly invertible strict 2-categories, and they claim that it easily extends to bi-groupoids. In the present work we will keep close to the first setting in order to generalize it to n -groupoids (weakly invertible strict n -categories).

1.6.1 Homotopy fibers

Let us suppose we are given a morphism (2-functor) of 2-groupoids

$$F : \mathbb{B} \rightarrow \mathbb{C}$$

that is a 2-functor between two 2-categories in which every arrow is an equivalence and every 2-cell is an isomorphism. If we fix an object b of \mathbb{B} , the homotopy fiber $\mathbb{F} = \mathbb{F}_{F, Fb}$ of F over Fb is the following 2-groupoid:

- objects are pairs $(b_0, Fb_0 \xrightarrow{c_1} Fb)$;
- an arrow $(b_0, c_1) \rightarrow (b'_0, c'_1)$ is a pair (b_1, c_2) as in the diagram below

$$\begin{array}{ccc} b_0 & \xrightarrow{b_1} & b'_0 \\ & \searrow c_1 & \nearrow c'_1 \\ & Fb & \end{array}$$

- a 2-cell $(b_1, c_2) \Rightarrow (b'_1, c'_2)$ is a pair (b_2, \equiv) as in the diagram below

$$\begin{array}{ccc} Fb_1 & \xRightarrow{Fb_2} & Fb'_1 \\ & \searrow c_2 & \nearrow c'_2 \\ & c_1 & \end{array}$$

The homotopy fiber comes with a 2-functor that embeds it in \mathbb{B}

$$K : \mathbb{F}_{F, Fb} \rightarrow \mathbb{B}$$

which sends the 2-cell

$$(b_2, \equiv) : (b_1, c_2) \Rightarrow (b'_1, c'_2) : (b_0, c_1) \rightarrow (b'_0, c'_1)$$

to $b_2 : b_1 \Rightarrow b'_1$.

1.6.2 2-Exact sequences

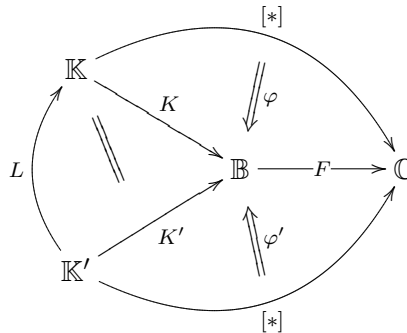
Before going further with our description, we urge to introduce a notion of exactness suitable for a 2-dimensional context.

A notion of *2-exactness* has been introduced by Vitale in [Vit02], in order to study some classical exact sequences of abelian groups associated with a morphism of commutative unital rings from sequences of pointed groupoids and categorical groups.

We report the definition in the context of pointed groupoids, as for categorical groups it applies plainly with no changes².

Let us consider the 2-category of pointed groupoids \mathbf{Gpd}_* , where the morphisms are functors that preserves the base points, 2-morphisms are natural isomorphisms whose component at the base point is the identity on the point. For a given morphism $F\mathbb{B} \rightarrow \mathbb{C}$, we define its *h-kernel* as the triple $(\mathbb{K}, K : \mathbb{K} \rightarrow \mathbb{B}, \varphi : [*] \Rightarrow KF)$ ($[*]$ denoting the constant 0-functor) satisfying the following universal property

Universal Property 1.1 (*h-kernels*). *For any other triple $(\mathbb{K}', K', \varphi')$ there exists a unique $L : \mathbb{K}' \rightarrow \mathbb{K}$ such that $K' = LK$.*



This universal property defines the (*h*-)kernel up to isomorphism.

²As a guiding analogy, do consider that exactness in the category of groups may be defined on the underlying pointed sets.

Remark 1.2. Let us notice that last universal property uses only whiskerings of morphisms with a 2-morphism, and does not use the full horizontal composition of 2-morphisms available in a 2-category. Hence a step forward towards a full generalization of Brown's result to weak n -structures can be accomplished by developing a theory that deals with these 1.5-universal properties (a.k.a. *sesqui-universal*, a.k.a. *h-universal*).

Remark 1.3. In dimension 1, our h -kernel satisfies also a universal property of a bi-limit, and this is indeed a point of view well considered in [DKV04]. Nevertheless, last *Remark* motivate the choice to restrict our attention to the h -limits considered.

Finally we are able to give the following

Definition 1.4. A triple (E, φ, F)

$$\begin{array}{ccccc} & & [*] & & \\ & \searrow & \Downarrow \varphi & \swarrow & \\ \mathbb{A} & \xrightarrow{E} & \mathbb{B} & \xrightarrow{F} & \mathbb{C} \end{array}$$

in \mathbf{Gpd}_* is called *exact* if the comparison with the 2-kernel is full and essentially surjective on objects.

Let us notice that in the above definition the 2-kernel can be replaced by the h -kernel, since fullness and essential surjectivity are preserved by equivalences.

We leave to the conscious reader the deepening of the theory of such exactness for the 2-dimensional context of pointed groupoids and categorical groups, w.r.t. cohomology, extensions, chain-complexes, ([Vit02, DKV04, BV02, Rou03, KV00, CGV06, KMV06, dRMMV05, GdR06, GDR05, GIdR04]).

1.6.3 Lowering the dimension: first step

Back to [DKV04], let us consider a morphism of 2-groupoids $F : \mathbb{B} \rightarrow \mathbb{C}$ and its homotopy kernel $\mathbb{K} \rightarrow \mathbb{B}$. They define indeed an exact sequence in the sesqui-category $2\mathbf{Gpd}_*$, but this point of view is not analyzed explicitly in [DKV04]. Instead the authors consider the diagram

$$\mathbb{K} \xrightarrow{K} \mathbb{B} \xrightarrow{F} \mathbb{C}$$

and they apply to that hom-of-the-point functor $[-]_1(*, *) : 2\mathbf{Gpd}_* \rightarrow \mathbf{Gpd}_*$ the classifying functor $\mathcal{C}\ell : 2\mathbf{Gpd}_* \rightarrow \mathbf{Gpd}_*$.

The first yields a 2-exact sequence

$$\mathbb{K}_1(*, *) \xrightarrow{K_1^{*,*}} \mathbb{B}_1(*, *) \xrightarrow{F_1^{*,*}} \mathbb{C}_1(*, *)$$

The second assigns to a 2-groupoid, the groupoid with the same set of objects, and whose arrows are 2-isomorphism classes of arrows [Bén67], thus providing the 2-exact sequence

$$\mathcal{C}\ell\mathbb{K} \xrightarrow{\mathcal{C}\ell K} \mathcal{C}\ell\mathbb{B} \xrightarrow{\mathcal{C}\ell F} \mathcal{C}\ell\mathbb{C}$$

Moreover it is possible to define a connecting functor

$$\delta : \mathbb{C}_1(*, *) \rightarrow \mathcal{C}\ell\mathbb{K}$$

such that the 6-term sequence obtained this way is everywhere 2-exact. The functor δ is defined as follows:

- (on objects) given an object $c_1 : * \rightarrow *$ in the domain,

$$\delta(c_1) = (*, c_1);$$

- (on arrows) given an arrow $c_2 : c_1 \Rightarrow c'_1$ in the domain,

$$\delta(c_2) = \{(1_*, c_2)\},$$

where brackets denote \mathcal{C} lasses (it is well defined).

1.6.4 Lowering the dimension: second step

The 6-term exact sequence

$$\mathbb{K}_1(*, *) \xrightarrow{K_1^{*,*}} \mathbb{B}_1(*, *) \xrightarrow{F_1^{*,*}} \mathbb{C}_1(*, *) \xrightarrow{\delta} \mathcal{C}\ell\mathbb{K} \xrightarrow{\mathcal{C}\ell K} \mathcal{C}\ell\mathbb{B} \xrightarrow{\mathcal{C}\ell F} \mathcal{C}\ell\mathbb{C}$$

with obvious transformations is such that the left-most three terms underly a strict monoidal structure given by the (former) 0-composition. Moreover, since we started with (weakly) invertible strict 2-categories, they are indeed categorical groups.

Now, if we denote by π_1 the functor $\mathbf{Gpd}_* \rightarrow \mathbf{Set}_*$ that assigns to a pointed groupoid the pointed set of the isomorphism classes of its objects, it is possible to show that it preserves exactness, *i.e.* it sends 2-exact sequences of pointed groupoids (categorical groups), to exact-sequences of pointed sets (groups). The same can be said of the functor π_0 .

Moreover $\pi_1(\mathcal{C}\ell(-)) = \pi_0([-]_1(*, *))$, hence we get a 9-term exact sequence

$$\begin{array}{ccccccc} \pi_1(\mathbb{K}_1(*, *)) & \longrightarrow & \pi_1(\mathbb{B}_1(*, *)) & \longrightarrow & \pi_1(\mathbb{C}_1(*, *)) & & \\ & & \swarrow & & \searrow & & \\ \pi_1(\mathcal{C}\ell\mathbb{K}) = \pi_0(\mathbb{K}_1(*, *)) & \longrightarrow & \pi_1(\mathcal{C}\ell\mathbb{B}) = \pi_0(\mathbb{B}_1(*, *)) & \longrightarrow & \pi_1(\mathcal{C}\ell\mathbb{C}) = \pi_0(\mathbb{C}_1(*, *)) & & \\ & & \swarrow & & \searrow & & \\ \pi_0(\mathcal{C}\ell\mathbb{K}) & \longrightarrow & \pi_0(\mathcal{C}\ell\mathbb{B}) & \longrightarrow & \pi_0(\mathcal{C}\ell\mathbb{C}) & & \end{array}$$

where the three left-most terms are abelian groups, the three central terms are groups, the three right-most terms are pointed sets. The reason why the three leftmost terms are abelian follows from a general fact of strict n -categories for homs over an identity cell (see [Sim98]), that is another *variazione* on the classical Eckmann-Hilton argument.

Chapter 2

Basics on sesqui-categories

2.1 Sesqui-categories

The notion of sesqui-category is due to Ross Street [Str96]. The term *sesqui* comes from the latin *semis-que*, that means (one and) a half. Hence a sesqui-category is something in-between a category and a 2-category. More precisely

Definition 2.1. A sesqui-category \mathcal{C} is a category $[\mathcal{C}]$ with a lifting of the hom-functor to \mathbf{Cat} , such that the following diagram of categories and functors commutes, \mathbf{obj} being the functor that forgets the morphisms:

$$\begin{array}{ccc} & & \mathbf{Cat} \\ & \nearrow \mathcal{C}(-,-) & \downarrow \mathbf{obj} \\ [\mathcal{C}]^{\mathrm{op}} \times [\mathcal{C}] & \xrightarrow{[\mathcal{C}](-,-)} & \mathbf{Set} \end{array} \quad (2.1)$$

Objects and morphisms of $[\mathcal{C}]$ are also objects and 1-cells of \mathcal{C} , while morphisms of $\mathcal{C}(A, B)$'s (with A and B running in $\mathbf{obj}([\mathcal{C}])$) are the 2-cells of \mathcal{C} .

We first observe that the definition above induces a 2-graph structure on \mathcal{C} , whose underlying graph underlies the category \mathcal{C} . Besides, the functor $\mathcal{C}(-, -)$ provides hom-sets of the category $[\mathcal{C}]$ with a category structure, whose composition is termed *vertical composition* (or 1-composition) of 2-cells. Finally, condition expressed by diagram (2.1) on the lifting $\mathcal{C}(-, -)$ gives a reduced horizontal composition, or *whiskering* (or 0-composition), compatible with 1-cell composition and with the 2-graph structure of \mathcal{C} .

In fact, for $A' \xrightarrow{a} A$ and $B \xrightarrow{b} B'$ in $[\mathcal{C}]$, the functor

$$\mathcal{C}(a, b) : \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A', B')$$

gives explicitly such a composition: for a 2-cell $\alpha : f \Rightarrow g : A \rightarrow B$, it whiskers the diagram

$$\begin{array}{ccccc} A' & \xrightarrow{a} & A & \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{array} & B & \xrightarrow{b} & B' \end{array}$$

to get the 2-cell

$$\begin{array}{ccc} A' & \begin{array}{c} \xrightarrow{a \bullet f \bullet b} \\ \Downarrow a \bullet \alpha \bullet b \\ \xrightarrow{a \bullet g \bullet b} \end{array} & B' \end{array}$$

where $a \bullet \alpha \bullet b$ is just a concise form for $\mathcal{C}(a, b)(\alpha)$.

By functoriality of whiskering, the operation may also be given in a *left-and-right* fashion. In fact it suffices to identify

$$a \bullet_L \alpha = a \bullet \alpha \bullet 1_B, \quad \alpha \bullet_R b = 1_A \bullet \alpha \bullet b$$

This fact can be made precise, and gives a more tractable definition, by the following characterization (see, for example [Gra94, Ste94]):

Proposition 2.2. *Let \mathcal{C} be a reflexive 2-graph*

$$\mathcal{C}_2 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{e} \\ \xrightarrow{t} \end{array} \mathcal{C}_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{e} \\ \xrightarrow{t} \end{array} \mathcal{C}_0$$

whose underlying graph $[\mathcal{C}] = \mathcal{C}_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{e} \\ \xrightarrow{t} \end{array} \mathcal{C}_0$ has a category structure. Then \mathcal{C} is a sesqui-category precisely when the following conditions hold:

1. for every pair of objects A, B of \mathcal{C}_0 , the graph $\mathcal{C}(A, B)$ has a category structure, called the hom-category of A, B .
2. (partial) reduced horizontal compositions are defined, i.e. for every A', A, B and B' objects of \mathcal{C}_0 , composition in $[\mathcal{C}]$ extends to binary operations

$$\bullet_L : [\mathcal{C}](A', A) \times \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A', B) \quad (2.2)$$

$$\bullet_R : \mathcal{C}(A, B) \times [\mathcal{C}](B, B') \longrightarrow \mathcal{C}(A, B'), \quad (2.3)$$

that satisfy equations below, whenever the composites are defined:

$$\begin{array}{ccccccc} A'' & \xrightarrow{a'} & A' & \xrightarrow{a} & A & \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{array} & B & \xrightarrow{b} & B' & \xrightarrow{b'} & A'' \end{array}$$

$$\begin{array}{ll}
(L1) & 1_A \bullet_L \alpha = \alpha \\
(L2) & a'a \bullet_L \alpha = a' \bullet_L (a \bullet_L \alpha) \\
(L3) & a \bullet_L 1_f = 1_{af} \\
(L4) & a \bullet_L (\alpha \cdot \beta) = (a \bullet_L \alpha) \cdot (a \bullet_L \beta) \\
(LR5) & (a \bullet_L \alpha) \bullet_R b = a \bullet_L (\alpha \bullet_R b)
\end{array}
\quad
\begin{array}{ll}
(R1) & \alpha \bullet_R 1_B = \alpha \\
(R2) & \alpha \bullet_R bb' = (\alpha \bullet_R b) \bullet_R b' \\
(R3) & 1_f \bullet_R b = 1_{fb} \\
(R4) & (\alpha \cdot \beta) \bullet_R b = (\alpha \bullet_R b) \cdot (\beta \bullet_R b)
\end{array}$$

In these equations, 1_A and 1_B are identity 1-cells, while 1_f , 1_{af} and 1_{fb} are identity 2-cells, and \cdot is the (vertical) composition inside the hom-categories. Axiom (LR5) will be also called whiskering axiom.

Proof. Let \mathcal{C} be a sesqui-category. The fact that $\mathcal{C}(A, B)$ are categories is clear from the definition, hence 1 is satisfied. Now, define for chosen A, A', B and B'

$$-1 \bullet_L -2 = \mathcal{C}(-1, 1_B)(-2) \quad -1 \bullet_R -2 = \mathcal{C}(1_A, -2)(-1)$$

Then for reduced left composition axioms, we have:

$$(L1) \quad 1_A \bullet_L \alpha = \mathcal{C}(1_A, 1_B)(\alpha) = \alpha$$

by functoriality w.r.t. units of $\mathcal{C}(-, -)$.

$$(L2) \quad (a'a) \bullet_L \alpha = \mathcal{C}(a'a, 1_B)(\alpha) = \mathcal{C}((a', 1_B)(a, 1_B))(\alpha) = \mathcal{C}(a', 1_B)(\mathcal{C}(a, 1_B)(\alpha))$$

by functoriality w.r.t. composition of $\mathcal{C}(-, -)$. Notice the contravariance on the first component.

$$(L3) \quad a \bullet_L 1_f = \mathcal{C}(a, 1_B)(1_f) = 1_{\mathcal{C}(a, 1_B)(f)} = 1_{af}$$

by functoriality w.r.t. units of $\mathcal{C}(a, 1_B)$.

$$(L4) \quad a \bullet_L (\alpha \cdot \beta) = \mathcal{C}(a, 1_B)(\alpha \cdot \beta) = \mathcal{C}(a, 1_B)(\alpha) \cdot \mathcal{C}(a, 1_B)(\beta) = (a \bullet_L \alpha) \cdot (a \bullet_L \beta)$$

by functoriality w.r.t. composition of $\mathcal{C}(a, 1_B)$.

Analogous proofs hold for (R1) to (R4). Finally

$$(LR5) \quad (a \bullet_L \alpha) \bullet_R b = \mathcal{C}(1_{A'}, b)(\mathcal{C}(a, 1_B)(\alpha)) = \mathcal{C}((1_{A'}, b)(a, 1_B))(\alpha) = \mathcal{C}(a, b)(\alpha) =$$

$$= \mathcal{C}((a, 1_{B'})(1_A, b))(\alpha) = \mathcal{C}(a, 1_{B'}) (\mathcal{C}(1_A, b)(\alpha)) = a \bullet_L (\alpha \bullet_R b)$$

by functoriality w.r.t. composition of $\mathcal{C}(-, -)$.

Conversely, supposing we are given a reflexive 2-graph \mathcal{C} underlying a category $[\mathcal{C}]$, hom-categories (1) and left/right-compositions satisfying conditions above (2). We show \mathcal{C} is a sesqui-category.

To this end we define a functor

$$[\mathcal{C}]^{\text{op}} \times [\mathcal{C}] \xrightarrow{\mathcal{C}(-, -)} \mathbf{Cat}$$

On objects, this is given by condition (1); on arrows, for

$$(a, b) : (A, B) \longrightarrow (A', B')$$

(i.e. $a : A' \longrightarrow A$ and $b : B \longrightarrow B'$), equation (2.5) helps us to define

$$\mathcal{C}(a, b) : \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A', B')$$

by

$$\mathcal{C}(a, b)(-) = a \bullet_L - \bullet_R b$$

These assignments give indeed functors and are functorial.

(i) $\mathcal{C}(a, b)$ is functor w.r.t. units

$$\mathcal{C}(a, b)(1_f) = a \bullet_L 1_f \bullet_R b = 1_{afb} = 1_{\mathcal{C}(a, b)(f)}$$

by (L3) and (R3)

(ii) $\mathcal{C}(a, b)$ is functor w.r.t. composition

$$\begin{aligned} \mathcal{C}(a, b)(\alpha \cdot \beta) &= a \bullet_L (\alpha \cdot \beta) \bullet_R b = ((a \bullet_L \alpha) \cdot (a \bullet_L \beta)) \bullet_R b = \\ &= (a \bullet_L \alpha \bullet_R b) \cdot (a \bullet_L \beta \bullet_R b) = \mathcal{C}(a, b)(\alpha) \cdot \mathcal{C}(a, b)(\beta) \end{aligned}$$

by (L4) and (R4).

(iii) $\mathcal{C}(-, -)$ is functor w.r.t. units

$$\mathcal{C}(1_A, 1_B)(\alpha) = 1_A \bullet_L \alpha \bullet_R 1_B = \alpha$$

by (L1) and (R1).

(vi) $\mathcal{C}(-, -)$ is functor w.r.t. composition

$$\begin{aligned} \mathcal{C}(a'a, bb')(\alpha) &= (a'a) \bullet_L \alpha \bullet_R (bb') = ((a' \bullet_L (a \bullet_L \alpha)) \bullet_R b) \bullet_R b' = \\ &= (a' \bullet_L \mathcal{C}(a, b)(\alpha)) \bullet_R b' = \mathcal{C}(a', b')(\mathcal{C}(a, b)(\alpha)) \end{aligned}$$

by (L2) and (R2).

That $\mathcal{C}(-, -)$ makes (2.1) commute is immediate from its definition. \square

Notice that reduced horizontal left/right composition will be often denoted simply by \bullet , when this does not cause ambiguity.

2.2 Morphisms of sesqui-categories

Morphisms between sesqui-categories are termed sesqui-functors. More precisely a sesqui-functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is a 2-graph morphism such that

- $\lfloor \mathcal{F} \rfloor : \lfloor \mathcal{C} \rfloor \longrightarrow \lfloor \mathcal{D} \rfloor$ is a functor,
- for every A, B in \mathcal{C}_0 ,

$$\mathcal{F}^{A,B} : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(\mathcal{F}(A), \mathcal{F}(B))$$

are functors component of a natural transformation \mathfrak{F}

$$\begin{array}{ccc} \lfloor \mathcal{C} \rfloor^{op} \times \lfloor \mathcal{C} \rfloor & & \\ \downarrow \lfloor \mathcal{F} \rfloor^{op} \times \lfloor \mathcal{F} \rfloor & \searrow \mathfrak{F} & \mathcal{C}(-, -) \\ \lfloor \mathcal{D} \rfloor^{op} \times \lfloor \mathcal{D} \rfloor & \nearrow \mathcal{D}(-, -) & \mathbf{Cat} \end{array} \quad (2.4)$$

that lifts $\lfloor \mathfrak{F} \rfloor : \lfloor \mathcal{C} \rfloor(-, -) \Rightarrow (\lfloor \mathcal{F} \rfloor^{op} \times \lfloor \mathcal{F} \rfloor) \cdot \lfloor \mathcal{D} \rfloor(-, -)$.

Remark 2.3. Notice that every functor between categories gives rise to such a natural transformation as $\lfloor \mathfrak{F} \rfloor$ for $\lfloor \mathcal{F} \rfloor$. From this point of view, the last condition may be re-formulated saying that *a sesqui-functor is the lifting of a functor between the underlying categories*.

We can translate the definition of sesqui-functor in terms of left/right compositions:

Proposition 2.4. *Let \mathcal{C} and \mathcal{D} be sesqui-categories, and let*

$$\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$$

be a 2-graphs homomorphism, whose underlying graph homomorphism

$$\lfloor \mathcal{F} \rfloor : \lfloor \mathcal{C} \rfloor \longrightarrow \lfloor \mathcal{D} \rfloor$$

is a functor.

Then \mathcal{F} is a sesqui-functor precisely when the following conditions hold:

1. *for every pair of objects A, B of \mathcal{C}_0 , the graph homomorphism*

$$\mathcal{F}^{A,B} : \mathcal{C}(A, B) \longrightarrow \mathcal{D}(\mathcal{F}(A), \mathcal{F}(B))$$

is a functor, called the hom-functor at A, B .

2. (partial) horizontal reduced compositions are preserved, i.e. for every diagram

$$\begin{array}{ccccc}
 A' & \xrightarrow{a} & A & \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{array} & B & \xrightarrow{b} & B'
 \end{array}$$

in \mathcal{C}_0 , equations below hold:

$$(L6) \quad \mathcal{F}(a \bullet_L \alpha) = \mathcal{F}(a) \bullet_L \mathcal{C}(\alpha) \quad (R6) \quad \mathcal{F}(\alpha \bullet_R b) = \mathcal{F}(\alpha) \bullet_R \mathcal{F}(b)$$

Proof. Let $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ be a sesqui-functor. Then \mathcal{F} is *a fortiori* a homomorphism of 2-graphs, underlying a functor $[\mathcal{F}]$. Furthermore, for every choice of A and B in \mathcal{C}_0 , the $\mathcal{F}^{A,B}$ are functors too. What remains to prove is that \mathcal{F} preserves left/right-compositions in the sense of (L6) and (R6), and this follows easily from naturality of \mathfrak{F} . In fact, for (L6)

$$\begin{aligned}
 \mathcal{F}(a \bullet_L \alpha) &= \mathcal{F}(\mathcal{C}(a, 1_B)(\alpha)) \text{ by definition} \\
 &= \mathcal{D}(\mathcal{F}(a), \mathcal{F}(1_B))(\mathcal{F}(\alpha)) \text{ by naturality} \\
 &= \mathcal{D}(\mathcal{F}(a), 1_{\mathcal{F}(B)})(\mathcal{F}(\alpha)) \text{ by functoriality} \\
 &= \mathcal{F}(a) \bullet_L \mathcal{F}(\alpha) \text{ by definition}
 \end{aligned}$$

(R6) from a similar calculation.

Conversely, suppose we are given two sesqui-categories \mathcal{C} and \mathcal{D} , together with a 2-graph homomorphism \mathcal{F} satisfying conditions (1) and (2) above.

We will prove naturality of \mathfrak{F} , i.e. for every $A' \xrightarrow{a} A$ and $B \xrightarrow{b} B'$ in \mathcal{C} , the following is a commutative diagram in **Cat**:

$$\begin{array}{ccc}
 \mathcal{C}(A, B) & \xrightarrow{\mathcal{C}(a,b)} & \mathcal{C}(A', B') \\
 \mathcal{F}^{A,B} \downarrow & & \downarrow \mathcal{F}^{A',B'} \\
 \mathcal{D}(\mathcal{F}(A), \mathcal{F}(B)) & \xrightarrow{\mathcal{D}(\mathcal{F}(a), \mathcal{F}(b))} & \mathcal{D}(\mathcal{F}(A'), \mathcal{F}(B'))
 \end{array}$$

That this diagram commutes on objects (i.e. on 1-cells of \mathcal{C} and \mathcal{D}) is clear from the fact that $[\mathcal{F}]$ is a functor and that left/right-compositions extend 1-cell-compositions. Finally, for a 2-cell α as above,

$$\begin{aligned}
 \mathcal{F}(\mathcal{C}(a, b)(\alpha)) &= \mathcal{F}(a \bullet_L \alpha \bullet_R b) \\
 &= \mathcal{F}(a \bullet_L \alpha) \bullet_R \mathcal{F}(b) \\
 &= \mathcal{F}(a) \bullet_L \mathcal{F}(\alpha) \bullet_R \mathcal{F}(b) \\
 &= \mathcal{D}(\mathcal{F}(a), \mathcal{F}(b))(\mathcal{F}(\alpha))
 \end{aligned}$$

follows from (L6) and (R6). □

Remark 2.5. In the following we will need the notion of 2-contravariant sesqui-functor. This is simply a sesqui-functor as above, such that the functors component $\mathcal{F}^{A,B}$ are usual contravariant functors.

Of course, characterization above still holds, *mutatis mutanda*: e.g. if

$$\alpha : f \Rightarrow g : A \rightarrow B,$$

then

$$\mathcal{F}(\alpha) : \mathcal{F}(g) \Rightarrow \mathcal{F}(f) : \mathcal{F}(A) \rightarrow \mathcal{F}(B).$$

2.3 2-Natural transformation of sesqui-functors

Definition 2.6 (strict sesqui-transformations). *Let two parallel sesqui-functors*

$$\mathcal{F}, \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$$

be given, and let be given a 2-graph transformation $\Delta : \mathcal{F} \Rightarrow \mathcal{G}$ whose underlying 1-transformation

$$[\Delta] : [\mathcal{F}] \Rightarrow [\mathcal{G}]$$

is a natural transformation of functors. Then Δ is a (strict) natural transformation of sesqui-functors when, for every $\alpha : f \Rightarrow g : A \rightarrow B$ in \mathcal{C} ,

$$\mathcal{F}(\alpha) \bullet_R \Delta_B = \Delta_A \bullet_L \mathcal{G}(\alpha)$$

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\Delta_A} & \mathcal{G}(A) \\ \mathcal{F}(f) \downarrow \scriptstyle \mathcal{F}(\alpha) \Rightarrow & & \downarrow \scriptstyle \mathcal{G}(\alpha) \Rightarrow \mathcal{G}(g) \\ \mathcal{F}(B) & \xrightarrow{\Delta_B} & \mathcal{G}(B) \end{array}$$

Notice that while vertical composition of (strict) natural transformation of sesqui-functors can be easily defined, the same is not true for horizontal composition. Therefore the category **SesquiCAT** of sesqui-categories, regardless of size issues, is indeed a sesqui-category itself.

The notion of (strict) natural transformation of sesqui-functors is essentially of a categorical nature. Namely the “functor”

$$[-] : \mathbf{SesquiCAT} \rightarrow \mathbf{CAT}$$

is also a “sesqui-functor”, when we consider the 2-category **CAT** as a sesqui-category.

Therefore those are just usual natural transformations that behave nice with respect to reduced left and right compositions. For the same reason the notions of adjunction and equivalence of sesqui-categories (w.r.t. strict transformations) are straightforward generalization of their categorical analogues.

Generalizing sesqui-categories (Chapter 5) we will need a further notion of sesqui-transformation, whose definition follows

Definition 2.7 (lax sesqui-transformations). *Let two parallel sesqui-functors*

$$\mathcal{F}, \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$$

be given, and let be given a 2-graph transformation $\Gamma : \mathcal{F} \Rightarrow \mathcal{G}$.

Then a lax natural transformation $\Gamma : \mathcal{G} \Rightarrow \mathcal{G}$ is given by the following data:

- *For every object A of \mathcal{C} , an arrow*

$$\Gamma_A : \mathcal{F}(A) \rightarrow \mathcal{G}(A)$$

- (naturality w.r.t. 1-cells) *For every arrow $f : A \rightarrow B$ of \mathcal{C} , a 2-cell*

$$\Gamma_f : \Gamma_A \bullet \mathcal{G}(f) \Rightarrow \mathcal{F}(f) \bullet \Gamma_B$$

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\Gamma_A} & \mathcal{G}(A) \\ \mathcal{F}(f) \downarrow & \swarrow \Gamma_f & \downarrow \mathcal{G}(f) \\ \mathcal{F}(B) & \xrightarrow{\Gamma_B} & \mathcal{G}(B) \end{array}$$

- (naturality w.r.t. 2-cells) *For every 2-cell $\alpha : f \Rightarrow g : A \rightarrow B$ in \mathcal{C} , an equation*

$$\begin{array}{ccc} \Gamma_A \bullet \mathcal{G}(f) & \xRightarrow{\Gamma_A \bullet_L \mathcal{G}(\alpha)} & \Gamma_A \bullet \mathcal{G}(g) \\ \Gamma_f \Downarrow & & \Downarrow \Gamma_g \\ \mathcal{F}(f) \bullet \Gamma_B & \xRightarrow{\mathcal{F}(\alpha) \bullet_R} & \mathcal{F}(g) \bullet \Gamma_B \end{array}$$

Those data have to satisfy the following functoriality axioms:

- For every object A of \mathcal{C}

$$\begin{array}{ccc}
 \mathcal{F}(A) & \xrightarrow{\Gamma_A} & \mathcal{G}(A) \\
 \mathcal{F}(1_A) \downarrow & \swarrow \Gamma_{1_A} & \downarrow \mathcal{G}(1_A) \\
 \mathcal{F}(A) & \xrightarrow{\Gamma_A} & \mathcal{G}(A)
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{F}(A) & \xrightarrow{\Gamma_A} & \mathcal{G}(A) \\
 1_{\mathcal{F}(A)} \downarrow & \swarrow 1_{\Gamma_A} & \downarrow 1_{\mathcal{G}(A)} \\
 \mathcal{F}(A) & \xrightarrow{\Gamma_A} & \mathcal{G}(A)
 \end{array}$$

- For every composable pair $A \xrightarrow{f} B \xrightarrow{h} C$ in \mathcal{C}

$$\begin{array}{ccc}
 \mathcal{F}(A) & \xrightarrow{\Gamma_A} & \mathcal{G}(A) \\
 \mathcal{F}(f) \downarrow & \swarrow \Gamma_f & \downarrow \mathcal{G}(f) \\
 \mathcal{F}(B) & \xrightarrow{\Gamma_B} & \mathcal{G}(B) \\
 \mathcal{F}(h) \downarrow & \swarrow \Gamma_h & \downarrow \mathcal{G}(h) \\
 \mathcal{F}(C) & \xrightarrow{\Gamma_C} & \mathcal{G}(C)
 \end{array}
 =
 \begin{array}{ccc}
 \mathcal{F}(A) & \xrightarrow{\Gamma_A} & \mathcal{G}(A) \\
 \mathcal{F}(fh) \downarrow & \swarrow \Gamma_{fh} & \downarrow \mathcal{G}(fh) \\
 \mathcal{F}(C) & \xrightarrow{\Gamma_C} & \mathcal{G}(C)
 \end{array}$$

Remark 2.8. In general, a lax sesqui-transformation is *not* a natural transformation of the functors underlying domain and co-domain sesqui-functors.

2.4 Sesqui-categories and 2-categories

That a sesqui-category induces a category structure on the underlying graph is clear from the very definition of sesqui-categories.

Hence, the question that naturally arises concerns *when* a sesqui-category is also a 2-category. In fact, given a sesqui-category \mathcal{C} , this underlies a 2-category precisely when, for every diagram of the kind

$$\begin{array}{ccccc}
 & f & & h & \\
 \bullet & \curvearrowright & \bullet & \curvearrowright & \bullet \\
 & \alpha & & \beta & \\
 & \downarrow & & \downarrow & \\
 & g & & k &
 \end{array}$$

the following equation is satisfied:

$$(f \bullet_L \beta) \cdot (\alpha \bullet_R k) = (\alpha \bullet_R h) \cdot (g \bullet_L \beta) \quad (2.5)$$

In this situation, the two composites are denoted $\alpha \bullet \beta$, and termed *horizontal composition* of α and β .

In other terms, it is possible to show that such a composition defines a family of functors

$$\mathcal{C}(A, B) \times \mathcal{C}(B, C) \xrightarrow{\bullet_{A,B,C}} \mathcal{C}(A, C)$$

indexed by triples (A, B, C) of objects of \mathcal{C} , satisfying 2-categorical axioms. There follows an interchange law for horizontal and vertical composition holds: for every four 2-cells $\alpha, \beta, \gamma, \delta$

$$\begin{array}{ccc} \bullet & \xrightarrow{\alpha \Downarrow \gamma} & \bullet \\ \alpha \Downarrow \gamma & & \beta \Downarrow \delta \\ \bullet & \xrightarrow{\beta \Downarrow \delta} & \bullet \end{array} \quad (\alpha \cdot \gamma) \bullet (\beta \cdot \delta) = (\alpha \bullet \beta) \cdot (\gamma \bullet \delta)$$

Even when equation (2.5) does not hold, some pasting operations of 2-cells are still available. We show this with an example.

Consider the diagram:

$$\begin{array}{ccccc} \bullet & \xrightarrow{a} & \bullet & \xrightarrow{b} & \bullet \\ d \downarrow & \alpha \swarrow & g \downarrow & \beta \swarrow & c \downarrow \\ \bullet & \xrightarrow{e} & \bullet & \xrightarrow{f} & \bullet \end{array}$$

Since intersection between α and β is one dimensional, it is unambiguous to define the *pasting*

$$(\alpha|\beta) = (a \bullet \beta) \cdot (\alpha \bullet f)$$

In the present work, we will not go any further into this subject.

2.5 Finite products in a sesqui-category

In the sesqui-categorical context we will refer to binary products according to the following 2-dimensional universal property

Definition 2.9. *Let \mathcal{C} be a sesqui-category, A and B two objects of \mathcal{C} . A product of A and B is a triple $(A \times B, \pi_A : A \times B \rightarrow A, \pi_B : A \times B \rightarrow B)$ satisfying the following*

Universal property

For every object Q of \mathcal{C} and 2-cells

$$\alpha : a \Rightarrow a' : Q \rightarrow A, \quad \beta : b \Rightarrow b' : Q \rightarrow B$$

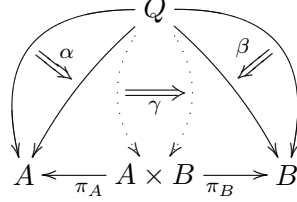
there exists a unique 2-cell

$$\gamma : q \Rightarrow q' : Q \rightarrow A \times B$$

with $\gamma \bullet \pi_A = \alpha$ and $\gamma \bullet \pi_B = \beta$.

We will write $\gamma = \langle \alpha, \beta \rangle$

The situation may be visualized on the diagram below



Such a product satisfies also the universal property defining categorical products. It suffices to choose $\alpha = 1_a$ and $\beta = 1_b$: the unique γ of the property satisfies $\gamma \bullet \pi_A = 1_a$ and $\gamma \bullet \pi_B = 1_b$. Hence, taking domains and codomains, we get

$$q\pi_A = a, \quad q\pi_B = b, \quad q'\pi_A = a, \quad q'\pi_B = b$$

This gives

$$\begin{aligned} 1_q \bullet \pi_A &= 1_{q\pi_A} \text{ by axiom (R3)} \\ &= 1_a \\ 1_q \bullet \pi_B &= 1_{q\pi_B} \text{ by axiom (L3)} \\ &= 1_b \end{aligned}$$

Finally, uniqueness forces $\gamma = 1_q$, and in turn, there exists a unique q ($= q'$) such that $q\pi_A = a$ and $q\pi_B = b$.

Definition 2.10. *Let \mathcal{C} be a sesqui-category. A terminal object is an object I of \mathcal{C} satisfying the following universal property*

(UP) *for every other object X of \mathcal{C} , there exists a unique 2-cell*

$$\xi : x \Rightarrow x' : X \rightarrow I$$

With a calculation similar to that of products, this universal property is equivalent to the existence of a unique $!_X : X \rightarrow I$, henceforth ξ is indeed the identity 2-cell on $!_X$.

Products and terminals defined this way are determined up to isomorphism. Furthermore finite products and canonical isomorphisms are defined as in the categorical case.

2.6 Product interchange rules

In the previous section we were concerned with properties of products in a sesqui-category that specialize in classical (viz. categorical) ones.

Now we focus our attention on products of 2-cells. What we recapture is the idea of independence of the components of a product, and a sort of commutativity that arises.

Consider the 2-cells $\alpha : f \Rightarrow g : A \rightarrow B$ and $\beta : h \Rightarrow k : C \rightarrow D$ in a sesqui-category \mathcal{C} . A 2-cell

$$\alpha \times \beta : f \times h \Rightarrow g \times k : A \times C \rightarrow B \times D$$

is uniquely determined by the universal property: $\alpha \times \beta = \langle \pi_A \bullet \alpha, \pi_C \bullet \beta \rangle$. Notice that this induces a kind of commutative horizontal composition of 2-cells, provided they are on different product-components.

In fact, we need the following

Lemma 2.11. *For α and β as above,*

$$((1_A \times \beta) \bullet (f \times 1_D)) \cdot ((1_A \times k) \bullet (\alpha \times 1_D)) = ((1_A \times h) \bullet (\alpha \times 1_D)) \cdot ((1_A \times \beta) \bullet (g \times 1_D))$$

Proof. By universality of products, they are both equal to $\alpha \times \beta$, because they have the same composite with projections. In fact we prove just the left side, the right side being analogous.

$$\begin{aligned} & ((1_A \times \beta) \bullet (f \times 1_D)) \cdot ((1_A \times k) \bullet (\alpha \times 1_D)) \bullet \pi_D = \\ &= ((1_A \times \beta) \bullet (f \times 1_D) \bullet \pi_D) \cdot (((1_A \times k) \bullet (\alpha \times 1_D)) \bullet \pi_D) \\ &= ((1_A \times \beta) \bullet \pi_D) \cdot ((1_A \times k) \bullet (\alpha \times 1_D) \bullet \pi_D) \\ &= ((1_A \times \beta) \bullet \pi_D) \cdot ((1_A \times k) \bullet \pi_D) \\ &= (\pi_C \bullet \beta) \cdot (\pi_C \bullet k) = \pi_C \bullet \beta \end{aligned}$$

$$\begin{aligned} & ((1_A \times \beta) \bullet (f \times 1_D)) \cdot ((1_A \times k) \bullet (\alpha \times 1_D)) \bullet \pi_B = \\ &= ((1_A \times \beta) \bullet (f \times 1_D) \bullet \pi_B) \cdot (((1_A \times k) \bullet (\alpha \times 1_D)) \bullet \pi_B) \\ &= ((1_A \times \beta) \bullet \pi_A f) \cdot ((1_A \times k) \bullet (\alpha \times 1_D) \bullet \pi_B) \\ &= (\pi_A f) \cdot ((1_A \times k) \pi_A \bullet \alpha) \\ &= (\pi_A f) \cdot (\pi_A \bullet \alpha) = \pi_A \bullet \alpha \end{aligned}$$

□

and to prove diagram equalities, such as the one above. These kind of diagrammatic equations will be called *product interchange rules*.

2.7 *h*-Pullbacks

We introduce here a notion of standard *h*-pullback suitable for our purposes. This notion has been formalized by Michael Mather in [Mat76b], for generic categories of spaces, with (eventually pointed) topological spaces in mind. It has been further generalized to *h*-categories¹ by Marco Grandis in [Gra94]. We (ab)use the term *h*-pullback, instead of that of *comma-square* because we will work mainly in a *n*-groupoidal context, with 2-morphisms being weakly invertible.

Definition 2.12. *Consider the following diagram in a sesqui-category \mathcal{C}*

$$\begin{array}{ccc} & C & \\ & \downarrow g & \\ A & \xrightarrow{f} & B \end{array}$$

An h -pullback of f and g is a four-tuple $(P(f, g), p, q, \varepsilon)$

$$\begin{array}{ccc} P & \xrightarrow{q} & C \\ p \downarrow & \nearrow \varepsilon & \downarrow g \\ A & \xrightarrow{f} & B \end{array}$$

where $P = P(f, g)$, that satisfies the following

Universal Property

For any other four-tuple (X, m, n, λ) as in

$$\begin{array}{ccc} X & \xrightarrow{n} & C \\ m \downarrow & \nearrow \lambda & \downarrow g \\ A & \xrightarrow{f} & B \end{array}$$

there exists a unique $\ell : X \rightarrow P$ such that

1. $\ell p = m$
2. $\ell q = n$

¹A *h*-category is a weaker notion than that of a sesqui-category, see [Gra94].

$$3. \ell \bullet_L \varepsilon = \lambda$$

Lemma 2.13. Universal Property 2.12 defines *h-pullbacks up to isomorphisms*.

Proof. Let (P, p, q, ε) be a *h-pullback*, according to definition above, and let $(P', p', q', \varepsilon')$ be another four-tuple satisfying the universal property. Then, since the first is a *h-pullback*, there exists $\ell : P' \rightarrow P$

$$\ell p = p', \quad \ell q = q', \quad \ell \varepsilon = \varepsilon'$$

and, since the second is a *h-pullback*, there exists $\ell' : P \rightarrow P'$

$$\ell' p' = p, \quad \ell' q' = q, \quad \ell' \varepsilon' = \varepsilon$$

Now, applying the universal property of the first one to itself, $\ell' \ell$ and id_P satisfy the same equations, and by uniqueness are equal. Similarly applying the universal property of the second one to itself, $\ell \ell'$ and $id_{P'}$ satisfy the same equations, and by uniqueness they are equal too. Hence ℓ and ℓ' are isomorphisms. \square

Lemma 2.14 (Pullback of *h-projections*). *In the sesqui-category \mathcal{C} , let be given the diagram below, where the left-hand square is commutative and the right-hand square ε is a *h-pullback**

$$\begin{array}{ccccc} R & \xrightarrow{s} & P & \xrightarrow{q} & D \\ r \downarrow & & p \downarrow & \nearrow \varepsilon & \downarrow g \\ A & \xrightarrow{e} & B & \xrightarrow{f} & C \end{array}$$

*then the composition $s \bullet_L \varepsilon$ is a *h-pullback* if, and only if, the left hand square is a pullback.*

Proof. We will show that the four-tuple $(R, r, sq, s \bullet_L \varepsilon)$ satisfies the universal property 6.1.

Let the four-tuple (X, y, z, ξ) be given as in the diagram below

$$\begin{array}{ccccc} X & \xrightarrow{z} & D \\ y \downarrow & & \nearrow \xi & & \downarrow g \\ A & \xrightarrow{e} & B & \xrightarrow{f} & C \end{array}$$

Since P is an *h-pullback*, there exists a unique $\ell : X \rightarrow P$ such that

$$(i) \ell p = ye, \quad (ii) \ell q = z, \quad (iii) \ell \bullet_L \varepsilon = \xi.$$

Yet since R is a pullback, condition (i) is equivalent to:

there exists a unique $x : X \rightarrow R$ such that

$$(iv) \quad xr = y, \quad (v) \quad xs = \ell.$$

Substituting, there exists a unique $x : X \rightarrow R$ such that

$$\begin{aligned} (i)' \quad & xr \underline{\underline{(iv)}} y \\ (ii)' \quad & xsq \underline{\underline{(v)}} \ell q \underline{\underline{(ii)}} z \\ (iii)' \quad & x \bullet_L (s \bullet_L \varepsilon) \underline{\underline{(L2)}} xs \bullet_L \varepsilon \underline{\underline{(v)}} \ell \bullet_L \varepsilon \underline{\underline{(iii)}} \xi \end{aligned}$$

□

Remark 2.15. This Lemma still holds in a mere *h*-category ([Gra94] *Lemma 2.2*).

Chapter 3

Strict n -categories

We give an inductive definition of the sesqui-category $n\mathbf{Cat}$, whose objects are (strict and small) n -categories, morphisms are (strict) n -functors and 2-morphisms are (lax) n -transformations. Furthermore, $n\mathbf{Cat}$ has sesqui-categorical finite products.

In the next three sections, we recall a standard inductive construction of $n\mathbf{Cat}$, well known in literature, recalled for instance in [Str87]. This is in fact a notion based upon a more general and influential theory of enrichment, developed by Gregory Maxwell Kelly (see [Kel05]).

This new perspective arises in the sesqui-categorical structure developed thereafter.

More precisely our notion of lax natural n -transformation generalizes the ordinary 2-dimensional version, as recalled for example in [Bor94]. This coincides with the inductive definition given in the internal abelian case in [Bou90]. Moreover it is equivalent to the global definition given in [Cra95] (see *Definition 9.1*, *Lemma 9.2* for a comparison), closely related to that of *m-fold homotopies* of ω -groupoids in [BH87].

For $n = 0$, $n\mathbf{Cat}$ is \mathbf{Set} , the sesqui-category of sets and maps with trivial (*i.e.* identity) transformations. Cartesian product provides the required sesqui-categorical product.

For $n = 1$, the 2-category \mathbf{Cat} of categories, functors and natural transformations has a underlying sesqui-categorical structure, when we consider only reduced horizontal composition of natural transformations with functors. Categorical product gives again the required sesqui-categorical product.

3.1 $n\mathbf{Cat}$: the data

n -categories

For given integer $n > 1$, a (strict) n -category \mathbb{C} consists of the following data:

- a set of objects \mathbb{C}_0 ;
- for every pair of objects c_0, c'_0 of \mathbb{C}_0 , a $(n-1)$ category

$$\mathbb{C}_1(c_0, c'_0)$$

called *hom* $(n-1)$ category over c_0 and c'_0 , and sometimes written $[c_0, c'_0]$ in order to simplify notation;

- for every object c_0 of \mathbb{C}_0 , a morphism of $(n-1)$ categories

$$\mathbb{I}_{(n-1)} \xrightarrow{u^0(c_0)} \mathbb{C}_1(c_0, c_0)$$

called the *0-identity* of c_0 ;

- for every triple of objects c_0, c'_0, c''_0 of \mathbb{C}_0 , a morphism of $(n-1)$ categories

$$\mathbb{C}_1(c_0, c'_0) \times_{(n-1)} \mathbb{C}_1(c'_0, c''_0) \xrightarrow{\circ_{c_0, c'_0, c''_0}^0} \mathbb{C}_1(c_0, c''_0)$$

called *0-composition*, following the *dimension-intersection* convention.

Here, $\times_{(n-1)}$ stays for the binary product in $(n-1)\mathbf{Cat}$, while $\mathbb{I}_{(n-1)}$ is the 0-ary product in $(n-1)\mathbf{Cat}$. Subscripts will be usually omitted, unless this causes confusion.

All these data must satisfy the following axioms, expressed by commutative diagrams in $(n-1)\mathbf{Cat}$:

- (*associativity axiom*) for every four-tuple of objects c_0, c'_0, c''_0, c'''_0 of \mathbb{C}_0

$$\begin{array}{ccc}
 (\mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c''_0)) \times \mathbb{C}_1(c''_0, c'''_0) & \xrightarrow{\alpha} & \mathbb{C}_1(c_0, c'_0) \times (\mathbb{C}_1(c'_0, c''_0) \times \mathbb{C}_1(c''_0, c'''_0)) \\
 \downarrow \circ_{c_0, c'_0, c''_0}^0 \times id & & \downarrow id \times \circ_{c'_0, c''_0, c'''_0}^0 \\
 \mathbb{C}_1(c_0, c''_0) \times \mathbb{C}_1(c''_0, c'''_0) & & \mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c'''_0) \\
 \searrow \circ_{c_0, c''_0, c'''_0}^0 & & \swarrow \circ_{c'_0, c''_0, c'''_0}^0 \\
 & \mathbb{C}_1(c_0, c'''_0) &
 \end{array} \tag{3.1}$$

where

$$\alpha = \alpha_{\mathbb{C}_1(c_0, c'_0), \mathbb{C}_1(c'_0, c''_0), \mathbb{C}_1(c''_0, c'''_0)}$$

is the usual associator given by universal property of product;

- (*left and right unit axioms*) for every pair of objects c_0, c'_0 of \mathbb{C}_0

$$\begin{array}{ccccc}
 \mathbb{I} \times \mathbb{C}_1(c_0, c'_0) & \xleftarrow[\sim]{\lambda} & \mathbb{C}_1(c_0, c'_0) & \xrightarrow[\sim]{\rho} & \mathbb{C}_1(c_0, c'_0) \times \mathbb{I} \\
 \downarrow u^0(c_0) \times id & & \parallel id & & \downarrow id \times u^0(c_0) \\
 \mathbb{C}_1(c_0, c_0) \times \mathbb{C}_1(c_0, c'_0) & \xrightarrow[\circ_{c_0, c_0, c'_0}^0]{} & \mathbb{C}_1(c_0, c'_0) & \xleftarrow[\circ_{c_0, c'_0, c'_0}^0]{} & \mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c'_0)
 \end{array} \tag{3.2}$$

where

$$\lambda = \lambda_{\mathbb{C}_1(c_0, c'_0)} \quad \text{and} \quad \rho = \rho_{\mathbb{C}_1(c_0, c'_0)}$$

are the usual left and right unit isomorphisms given by the universal property of the product.

Morphisms of *n*-categories

For a given integer $n > 1$, and given *n*-categories \mathbb{C} and \mathbb{D} , a (strict) *n*-functor

$$F : \mathbb{C} \longrightarrow \mathbb{D}$$

is a pair (F_0, F_1) where:

- $F_0 : \mathbb{C}_0 \longrightarrow \mathbb{D}_0$ is a map;
- for every pair of objects c_0, c'_0 of \mathbb{C}_0

$$F_1^{c_0, c'_0} : \mathbb{C}_1(c_0, c'_0) \longrightarrow \mathbb{D}_1(F_0 c_0, F_0 c'_0)$$

is a morphism of $(n-1)$ categories.

These data must satisfy the following axioms, expressed by commutative diagrams in $(n-1)\mathbf{Cat}$:

- (*functoriality w.r.t. composition*) for every triple of objects c_0, c'_0, c''_0 of \mathbb{C}_0

$$\begin{array}{ccc}
 \mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c''_0) & \xrightarrow{\circ_{c'_0}^0} & \mathbb{C}_1(c_0, c''_0) \\
 \downarrow F_1^{c_0, c'_0} \times F_1^{c'_0, c''_0} & & \downarrow F_1^{c_0, c''_0} \\
 \mathbb{D}_1(F_0 c_0, F_0 c'_0) \times \mathbb{D}_1(F_0 c'_0, F_0 c''_0) & \xrightarrow{\circ_{F_0 c'_0}^0} & \mathbb{D}_1(F_0 c_0, F_0 c''_0)
 \end{array} \tag{3.3}$$

- (*functoriality w.r.t. units*) for every triple of object c_0 of \mathbb{C}_0

$$\begin{array}{ccc}
 \mathbb{I} & \xrightarrow{c_{u^0}(c_0)} & \mathbb{C}_1(c_0, c_0) \\
 & \searrow \mathbb{D}_{u^0}(F_0 c_0) & \downarrow F_1^{c_0, c_0} \\
 & & \mathbb{D}_1(F_0 c_0, F_0 c_0)
 \end{array} \tag{3.4}$$

2-Morphisms of n -categories

For a given integer $n > 1$, and given n -functors $F, G : \mathbb{C} \longrightarrow \mathbb{D}$, a lax natural n -transformation

$$\begin{array}{ccc}
 & F & \\
 \mathbb{C} & \begin{array}{c} \curvearrowright \\ \Downarrow \alpha \\ \curvearrowleft \end{array} & \mathbb{D} \\
 & G &
 \end{array}$$

is a pair $\alpha = (\alpha_0, \alpha_1)$, where

- $\alpha_0 : \mathbb{C}_0 \longrightarrow \coprod_{c_0 \in \mathbb{C}_0} [\mathbb{D}_1(F_0 c_0, G_0 c_0)]_0$ is a map such that, for every c_0 in \mathbb{C}_0 , $\alpha_0(c_0) : F_0(c_0) \longrightarrow G_0(c_0)$;
- (n -*naturality*) for every pair c_0, c'_0 of \mathbb{C} , $\alpha_1^{c_0, c'_0}$ is a 2-morphism of $(n-1)\mathbf{Cat}$, as in the diagram below

$$\begin{array}{ccccc}
 & & \mathbb{C}_1(c_0, c'_0) & & \\
 & \swarrow F_1^{c_0, c'_0} & & \searrow G_1^{c_0, c'_0} & \\
 \mathbb{D}_1(F_0 c_0, F_0 c'_0) & & \xleftarrow{\alpha_1^{c_0, c'_0}} & & \mathbb{D}_1(G_0 c_0, G_0 c'_0) \\
 & \searrow -\circ^0 \alpha_0 c'_0 & & \swarrow \alpha_0 c_0 \circ^0 - & \\
 & & \mathbb{D}_1(F_0 c_0, G_0 c'_0) & &
 \end{array}$$

In order to keep notation lighter we will often write α_{c_0} instead of $\alpha_0(c_0)$. These data must satisfy functoriality axioms expressed by the following equations of diagrams in $(n-1)\mathbf{Cat}$:

- (*functoriality w.r.t. composition*) for every triple of objects c_0, c'_0, c''_0 of \mathbb{C}_0 ,

$$\begin{array}{c}
\mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c''_0) \\
\swarrow \text{id} \times F_1^{c'_0, c''_0} \quad \searrow G_1^{c'_0, c''_0} \times \text{id} \\
\mathbb{C}_1(c_0, c'_0) \times \mathbb{D}_1(F_0 c'_0, F_0 c''_0) \quad \mathbb{D}_1(G_0 c_0, G_0 c'_0) \times \mathbb{C}_1(c'_0, c''_0) \\
\downarrow \text{id} \times (-\circ \alpha_0 c'_0) \quad \downarrow \alpha_1^{c'_0, c''_0} \times \text{id} \quad \downarrow (\alpha_0 c_0 \circ -) \times \text{id} \\
\mathbb{C}_1(c_0, c'_0) \times \mathbb{D}_1(F_0 c'_0, G_0 c'_0) \quad \mathbb{C}_1(c_0, c'_0) \times \mathbb{D}_1(G_0 c'_0, G_0 c''_0) \quad \mathbb{D}_1(F_0 c_0, F_0 c'_0) \times \mathbb{C}_1(c'_0, c''_0) \quad \mathbb{D}_1(F_0 c_0, G_0 c'_0) \times \mathbb{C}_1(c'_0, c''_0) \\
\downarrow F_1^{c'_0, c''_0} \times \text{id} \quad \downarrow \text{id} \times G_1^{c'_0, c''_0} \quad \downarrow (-\circ \alpha_0 c'_0) \times \text{id} \quad \downarrow \text{id} \times G_1^{c'_0, c''_0} \\
\mathbb{D}_1(F_0 c_0, F_0 c'_0) \times \mathbb{D}_1(F_0 c'_0, G_0 c'_0) \quad \mathbb{D}_1(F_0 c_0, G_0 c'_0) \times \mathbb{D}_1(G_0 c'_0, G_0 c''_0) \\
\searrow \circ^0 \quad \swarrow \circ^0 \\
\mathbb{D}_1(F_0 c_0, G_0 c''_0)
\end{array}
\tag{3.5}$$

$$\begin{array}{c}
\mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c''_0) \\
\downarrow \circ^0 \\
\mathbb{C}_1(c_0, c''_0) \\
\swarrow F_1^{c'_0, c''_0} \quad \searrow G_1^{c'_0, c''_0} \\
\mathbb{D}_1(F_0 c_0, F_0 c''_0) \quad \mathbb{D}_1(G_0 c_0, G_0 c''_0) \\
\swarrow -\circ \alpha_0 c'_0 \quad \searrow \alpha_0 c_0 \circ - \\
\mathbb{D}_1(F_0 c_0, G_0 c''_0)
\end{array}
=$$

- (functoriality w.r.t. units) for every object c_0 of \mathbb{C}_0 ,

$$\begin{array}{c}
\text{II} \\
\downarrow u^0(c_0) \\
\mathbb{C}_1(c_0, c_0) \\
\swarrow F_1^{c_0, c_0} \quad \searrow G_1^{c_0, c_0} \\
\mathbb{D}_1(F_0 c_0, F_0 c_0) \quad \mathbb{D}_1(G_0 c_0, G_0 c_0) \\
\swarrow -\circ \alpha_0 c_0 \quad \searrow \alpha_0 c_0 \circ - \\
\mathbb{D}_1(F_0 c_0, G_0 c_0)
\end{array}
=
\begin{array}{c}
\text{II} \\
\downarrow [\alpha_0 c_0] \quad \downarrow [\alpha_0 c_0] \\
\mathbb{D}_1(F_0 c_0, G_0 c_0)
\end{array}
\tag{3.6}$$

Remark 3.1. In defining transformations, we used expressions such as $-\circ \alpha_{c'_0}$ or $\alpha_{c_0} \circ -$ to denote composition (n-1)functors. In fact, given a n-category \mathbb{C}

and a 1-cell $c_1 : c_0 \rightarrow c'_0$, it is always possible to define a pair of $(n-1)$ functors for every other chosen objects \bar{c}_0 of \mathbb{C} :

$$\begin{aligned} [- \circ^0 c_1]_{\bar{c}_0} : \mathbb{C}_1(\bar{c}_0, c_0) &\rightarrow \mathbb{C}_1(\bar{c}_0, c'_0) \\ [c_1 \circ^0 -]_{\bar{c}_0} : \mathbb{C}_1(c'_0, \bar{c}_0) &\rightarrow \mathbb{C}_1(c_0, \bar{c}_0) \end{aligned}$$

As a matter of fact, they are just restrictions of 0-composition functors in \mathbb{C} . The assignment is (mutually) natural in \bar{c}_0 .

In fact, whiskering makes the following diagram commute, for $\bar{c}_1 : \bar{\bar{c}}_0 \rightarrow \bar{c}_0$:

$$\begin{array}{ccc} \mathbb{C}_1(\bar{c}_0, c_0) & \xrightarrow{- \circ c_1} & \mathbb{C}_1(\bar{c}_0, c'_0) \\ \bar{c}_1 \circ - \downarrow & & \downarrow \bar{c}_1 \circ - \\ \mathbb{C}_1(\bar{\bar{c}}_0, c_0) & \xrightarrow{- \circ c_1} & \mathbb{C}_1(\bar{\bar{c}}_0, c'_0) \end{array}$$

A natural n -transformation of n -functors $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$ is called strict when for every pair of objects c_0, c'_0 of \mathbb{C} , $\alpha_1^{c_0, c'_0}$ is an identity.

3.2 $n\mathbf{Cat}$: the underlying category

$n\mathbf{Cat}$ has the underlying category denoted $[n\mathbf{Cat}]$, whose description follows. Notice that in this section we will assume n be an integer greater than 1.

Given n -functors

$$\mathbb{C} \xrightarrow{F} \mathbb{D} \xrightarrow{G} \mathbb{E}$$

their composition $F \bullet^0 G$ (or simply FG) is the morphism with

$$[F \bullet^0 G]_0 = F_0 G_0$$

in \mathbf{Set} , and, for every pair of objects c_0, c'_0 of \mathbb{C}_0 ,

$$[F \bullet^0 G]_1^{c_0, c'_0} = F_1^{c_0, c'_0} \bullet^0 G_1^{F_0 c_0, F_0 c'_0}$$

in $(n-1)\mathbf{Cat}$.

The pair $([F \bullet^0 G]_0, [F \bullet^0 G]_1)$ defines indeed a morphism of $(n-1)\mathbf{Cat}$. This is clear by pasting the commutative diagrams below, for every triple of objects c_0, c'_0, c''_0 of \mathbb{C}_0 . They ensure functoriality w.r.t. composition and units of (3.3) and (3.6).

$$\begin{array}{ccc} [c_0, c'_0] \times [c'_0, c''_0] & \xrightarrow{c_0 \circ} & [c_0, c''_0] \\ \downarrow F_1^{c_0, c'_0} \times F_1^{c'_0, c''_0} & & \downarrow F_1^{c_0, c''_0} \\ [Fc_0, Fc'_0] \times [Fc'_0, Fc''_0] & \xrightarrow{D_0 \circ} & [Fc_0, Fc''_0] \\ \downarrow G_1^{Fc_0, Fc'_0} \times G_1^{Fc'_0, Fc''_0} & & \downarrow G_1^{Fc_0, Fc''_0} \\ [G(Fc_0), G(Fc'_0)] \times [G(Fc'_0), G(Fc''_0)] & \xrightarrow{E_0 \circ} & [G(Fc_0), G(Fc''_0)] \end{array} \quad \begin{array}{ccc} \mathbb{I} & \xrightarrow{c_u^0(c_0)} & [c_0, c_0] \\ & \searrow D_u^0(Fc_0) & \downarrow F_1^{c_0, c_0} \\ & \searrow E_u^0(G(Fc_0)) & [Fc_0, Fc_0] \\ & & \downarrow G_1^{Fc_0, Fc_0} \\ & & [G(Fc_0), G(Fc_0)] \end{array}$$

Furthermore, for every n -category \mathbb{C} , an identity functor

$$\mathbb{C} \xrightarrow{id_{\mathbb{C}}} \mathbb{C}$$

is defined by the pair $([id_{\mathbb{C}}]_0, [id_{\mathbb{C}}]_1)$, where

$$[id_{\mathbb{C}}]_0 = id_{\mathbb{C}_0}$$

in \mathbf{Set} , and, for every pair of objects c_0, c'_0 of \mathbb{C}

$$[id_{\mathbb{C}}]_1^{c_0, c'_0} = id_{\mathbb{C}_1(c_0, c'_0)}$$

in $(n-1)\mathbf{Cat}$.

Notice that $([id_{\mathbb{C}}]_0, [id_{\mathbb{C}}]_1)$ satisfies trivially functoriality diagrams (3.3) and (3.6).

Proposition 3.2. *(small and strict) n -categories and (strict) n -functors define a category: $[n\mathbf{Cat}]$*

Proof. Firstly, for every pair \mathbb{C}, \mathbb{D} of (small) n -categories,

$$Hom(\mathbb{C}, \mathbb{D}) = \{n\text{-functors } F : \mathbb{C} \rightarrow \mathbb{D}\}$$

is a set, since \mathbb{D} is small.

Hence we will show that composition is associative and that identities are neutral.

Concerning associativity, let a composable triple of morphisms be given:

$$\mathbb{C} \xrightarrow{F} \mathbb{D} \xrightarrow{G} \mathbb{E} \xrightarrow{H} \mathbb{F}$$

We want to prove $(FG)H = F(GH)$.

On objects, let us consider the following equalities in the category \mathbf{Set} :

$$\begin{aligned} [(FG)H]_0 &= [FG]_0 H_0 = \\ &= (F_0 G_0) H_0 = \\ &= F_0 (G_0 H_0) = \\ &= F_0 [GH]_0 = [F(GH)]_0 \end{aligned}$$

Besides, for every pair of objects c_0, c'_0 of \mathbb{C} , $(n-1)$ -associativity implies:

$$\begin{aligned} [(FG)H]_1^{c_0, c'_0} &= [FG]_1^{c_0, c'_0} H_1^{G(Fc_0), G(Fc'_0)} = \\ &= \left(F_1^{c_0, c'_0} G_1^{Fc_0, Fc'_0} \right) H_1^{G(Fc_0), G(Fc'_0)} = \\ &= F_1^{c_0, c'_0} \left(G_1^{Fc_0, Fc'_0} H_1^{G(Fc_0), G(Fc'_0)} \right) = \\ &= F_1^{c_0, c'_0} [GH]_1^{Fc_0, Fc'_0} = [F(GH)]_1^{c_0, c'_0} \end{aligned}$$

Turning now to identities, let us consider the situation:

$$\mathbb{C} \xrightarrow{id_{\mathbb{C}}} \mathbb{C} \xrightarrow{F} \mathbb{D} \xrightarrow{id_{\mathbb{D}}} \mathbb{D}$$

We want to prove $id_{\mathbb{C}}F = F = F id_{\mathbb{D}}$. On objects,

$$\begin{aligned} [id_{\mathbb{C}}F]_0 &= [id_{\mathbb{C}}]_0 F_0 = \\ &= id_{\mathbb{C}_0} F_0 = \\ &= F_0 = F_0 id_{\mathbb{D}_0} = \\ &= F_0 [id_{\mathbb{D}}]_0 = [F id_{\mathbb{D}}]_0 \end{aligned}$$

and, for every pair of objects c_0, c'_0 of \mathbb{C} , neutral $(n-1)$ -identities imply:

$$\begin{aligned} [id_{\mathbb{C}}F]_1^{c_0, c'_0} &= [id_{\mathbb{C}}]_1^{c_0, c'_0} F_1^{c_0, c'_0} = \\ &= id_{\mathbb{C}_1(c_0, c'_0)} F_1^{c_0, c'_0} = \\ &= F_1^{c_0, c'_0} = F_1^{c_0, c'_0} id_{\mathbb{D}_1(Fc_0, Fc'_0)} = \\ &= F_1^{c_0, c'_0} [id_{\mathbb{D}}]_1^{Fc_0, Fc'_0} = [F id_{\mathbb{D}}]_1^{c_0, c'_0} \end{aligned}$$

□

Proposition 3.3. *The category $[n\mathbf{Cat}]$ has finite products.*

Proof. We will show that $[\mathbf{Cat}]$ has a terminal object and binary products. Given n -categories \mathbb{C} and \mathbb{D} , their (standard) product is defined as follows:

$$[\mathbb{C} \times \mathbb{D}]_0 = \mathbb{C}_0 \times \mathbb{D}_0$$

and, for every pair $(c_0, d_0), (c'_0, d'_0)$ in $[\mathbb{C} \times \mathbb{D}]_0$,

$$[\mathbb{C} \times \mathbb{D}]_1((c_0, d_0), (c'_0, d'_0)) = \mathbb{C}_1(c_0, c'_0) \times \mathbb{D}_1(d_0, d'_0)$$

Composition is defined by means of universality of products in $(n-1)\mathbf{Cat}$: for every triple $(c_0, d_0), (c'_0, d'_0)$ and (c''_0, d''_0) in $[\mathbb{C} \times \mathbb{D}]_0$, the dotted arrow in the diagram below gives composition:

$$\begin{array}{ccc} [\mathbb{C} \times \mathbb{D}]_1((c_0, d_0), (c'_0, d'_0)) \times [\mathbb{C} \times \mathbb{D}]_1((c'_0, d'_0), (c''_0, d''_0)) & \xrightarrow{\quad \mathbb{C} \times \mathbb{D}_o \quad} & [\mathbb{C} \times \mathbb{D}]_1((c_0, d_0), (c''_0, d''_0)) \\ \parallel \scriptstyle id & & \parallel \scriptstyle id \\ \left(\mathbb{C}_1(c_0, c'_0) \times \mathbb{D}_1(d_0, d'_0) \right) \times \left(\mathbb{C}_1(c'_0, c''_0) \times \mathbb{D}_1(d'_0, d''_0) \right) & & \\ \downarrow \scriptstyle \tau & & \\ \left(\mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c''_0) \right) \times \left(\mathbb{D}_1(d_0, d'_0) \times \mathbb{D}_1(d'_0, d''_0) \right) & \xrightarrow{\quad \mathbb{C}_o \times \mathbb{D}_o \quad} & \mathbb{C}_1(c_0, c''_0) \times \mathbb{D}_1(d_0, d''_0) \end{array}$$

where the twist isomorphism $\tau = \tau_{\mathbb{C}_1(c_0, c'_0), \mathbb{D}_1(d_0, d'_0), \mathbb{C}_1(c'_0, c''_0), \mathbb{D}_1(d'_0, d''_0)}$ is given by products properties in $(n-1)\mathbf{Cat}$.

Identities are defined in the same way. For every object (c_0, d_0) in $\mathbb{C} \times \mathbb{D}$, by the dotted arrow in the diagram below:

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{\mathbb{C} \times \mathbb{D} u^0((c_0, d_0))} & [\mathbb{C} \times \mathbb{D}]_1((c_0, d_0), (c_0, d_0)) \\ \text{\textit{iso}} \downarrow & & \parallel \text{\textit{id}} \\ \mathbb{I} \times \mathbb{I} & \xrightarrow{c u^0(c_0) \times d u^0(d_0)} & \mathbb{C}_1(c_0, c_0) \times \mathbb{D}_1(d_0, d_0) \end{array}$$

Product projections

$$\mathbb{C} \xleftarrow{\Pi_{\mathbb{C}}} \mathbb{C} \times \mathbb{D} \xrightarrow{\Pi_{\mathbb{D}}} \mathbb{D}$$

are given respectively by projections

$$[\Pi_{\mathbb{C}}]_0 = \pi_{\mathbb{C}_0}^{\mathbb{C}_0 \times \mathbb{D}_0}, \quad [\Pi_{\mathbb{D}}]_0 = \pi_{\mathbb{D}_0}^{\mathbb{C}_0 \times \mathbb{D}_0}$$

and by the following compositions in $(n-1)\mathbf{Cat}$, for every pair (c_0, d_0) and (c'_0, d'_0) :

$$\begin{array}{ccc} [\mathbb{C} \times \mathbb{D}]_1((c_0, d_0), (c'_0, d'_0)) & \xrightarrow{[\Pi_{\mathbb{C}}]_1^{(c_0, d_0), (c'_0, d'_0)}} & \mathbb{C}_1(c_0, c'_0) \\ & \searrow \text{\textit{id}} \times ! \quad \nearrow \text{\textit{iso}} & \\ & \mathbb{C}_1(c_0, c'_0) \times \mathbb{I} & \end{array}$$

$$\begin{array}{ccc} [\mathbb{C} \times \mathbb{D}]_1((c_0, d_0), (c'_0, d'_0)) & \xrightarrow{[\Pi_{\mathbb{D}}]_1^{(c_0, d_0), (c'_0, d'_0)}} & \mathbb{D}_1(d_0, d'_0) \\ & \searrow \text{\textit{id}} \times ! \quad \nearrow \text{\textit{iso}} & \\ & \mathbb{D}_1(d_0, d'_0) \times \mathbb{I} & \end{array}$$

It is a matter of repeated use of the universal property of products in $(n-1)\mathbf{Cat}$ to prove that all these data define a n -category and two n -categories morphisms, and that $(\mathbb{C} \times \mathbb{D}, \Pi_{\mathbb{C}}, \Pi_{\mathbb{D}})$ is a product in $n\mathbf{Cat}$.

Concerning the terminal n -category, a standard construction follows. A terminal $\mathbb{I}_{(n)}$ is given by the pair

$$[\mathbb{I}_{(n)}]_0 = \{*\}, \quad [\mathbb{I}_{(n)}]_0^{*,*} = \mathbb{I}_{(n-1)};$$

with composition ${}_{(n-1)}\mathbb{I}_{(n)} \times {}_{(n-1)}\mathbb{I}_{(n)} \xrightarrow{\sim} {}_{(n-1)}\mathbb{I}_{(n)}$.

Classical constructions of categorical limits help in defining n -ary products and canonical isomorphisms

$$\alpha_{\mathbb{A}, \mathbb{B}, \mathbb{C}} : (\mathbb{A} \times \mathbb{B}) \times \mathbb{C} \xrightarrow{\sim} \mathbb{A} \times (\mathbb{B} \times \mathbb{C}) \quad (3.7)$$

$$\rho_{\mathbb{A}} : \mathbb{A} \xrightarrow{\sim} \mathbb{A} \times_{(\mathbb{I}_n)} \mathbb{I}_n \quad (3.8)$$

$$\lambda_{\mathbb{A}} : \mathbb{A} \xrightarrow{\sim} \mathbb{I}_n \times \mathbb{A} \quad (3.9)$$

$$\tau_{\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}} : (\mathbb{A} \times \mathbb{B}) \times (\mathbb{C} \times \mathbb{D}) \xrightarrow{\sim} (\mathbb{A} \times \mathbb{C}) \times (\mathbb{B} \times \mathbb{D}) \quad (3.10)$$

□

3.3 $n\mathbf{Cat}$: the hom-categories

In this section we describe, hom-categories $n\mathbf{Cat}(\mathbb{C}, \mathbb{D})$, once n -categories \mathbb{C} and \mathbb{D} are fixed.

3.3.1 Vertical composition

Given the diagram:

$$\begin{array}{ccc} & E & \\ \swarrow & \Downarrow \omega & \searrow \\ \mathbb{C} & \xrightarrow{F} & \mathbb{D} \\ \nwarrow & \Downarrow \alpha & \nearrow \\ & G & \end{array} ;$$

one defines a (vertical, or 1-)composition $\omega \bullet^1 \alpha : E \Rightarrow G$ in the following way:

- for every object c_0 in \mathbb{C} ,

$$[\omega \alpha]_0(c_0) = \omega_0 c_0 \circ^0 \alpha_0 c_0 : E c_0 \longrightarrow G c_0$$

- for every pair of objects c_0, c'_0 in \mathbb{C} , the diagram below describes

$$[\omega \bullet^1 \alpha]_1^{c_0, c'_0} = \left(\omega_{c_0} \circ \alpha_1^{c_0, c'_0} \right) \bullet^1 \left(\omega_1^{c_0, c'_0} \circ \alpha_{c'_0} \right)$$

$$\begin{array}{ccccc}
& \mathbb{C}_1(c_0, c'_0) & & & \\
& \swarrow E_1 & \downarrow F_1 & \searrow G_1 & \\
\mathbb{D}_1(Ec_0, Ec'_0) & & \mathbb{D}_1(Fc_0, Fc'_0) & & \mathbb{D}_1(Gc_0, Gc'_0) \\
& \swarrow \omega_1^{c_0, c'_0} & & \swarrow \alpha_1^{c_0, c'_0} & \\
& \mathbb{D}_1(Ec_0, Fc'_0) & \equiv & \mathbb{D}_1(Fc_0, Gc'_0) & \\
& \swarrow \omega c_0 \circ - & & \swarrow - \circ \alpha c'_0 & \\
& \mathbb{D}_1(Ec_0, Gc'_0) & & &
\end{array}$$

$\begin{array}{ccc} \downarrow - \circ \omega c'_0 & & \downarrow \alpha c_0 \circ - \\ \downarrow - \circ \alpha c'_0 & & \downarrow \omega c_0 \circ - \end{array}$

To prove that these data give indeed a 2-morphism, unit functoriality (3.6) and composition functoriality (3.5) equations must hold.

To this end, chose an object c_0 of \mathbb{C} and consider the following chain of diagrams equalities:

$$\begin{array}{ccccc}
& \mathbb{I} & & & \\
& \downarrow u(c_0) & & & \\
& [c_0, c_0] & & & \\
& \swarrow E_1 & \downarrow F_1 & \searrow G_1 & \\
[Ec_0, Ec_0] & & [Fc_0, Fc_0] & & [Gc_0, Gc_0] \\
& \swarrow \omega_1^{c_0, c_0} & & \swarrow \alpha_1^{c_0, c_0} & \\
& [Ec_0, Fc_0] & \equiv & [Fc_0, Gc_0] & \\
& \swarrow \omega c_0 \circ - & & \swarrow - \circ \alpha c_0 & \\
& [Ec_0, Gc_0] & & &
\end{array}
\quad (i)$$

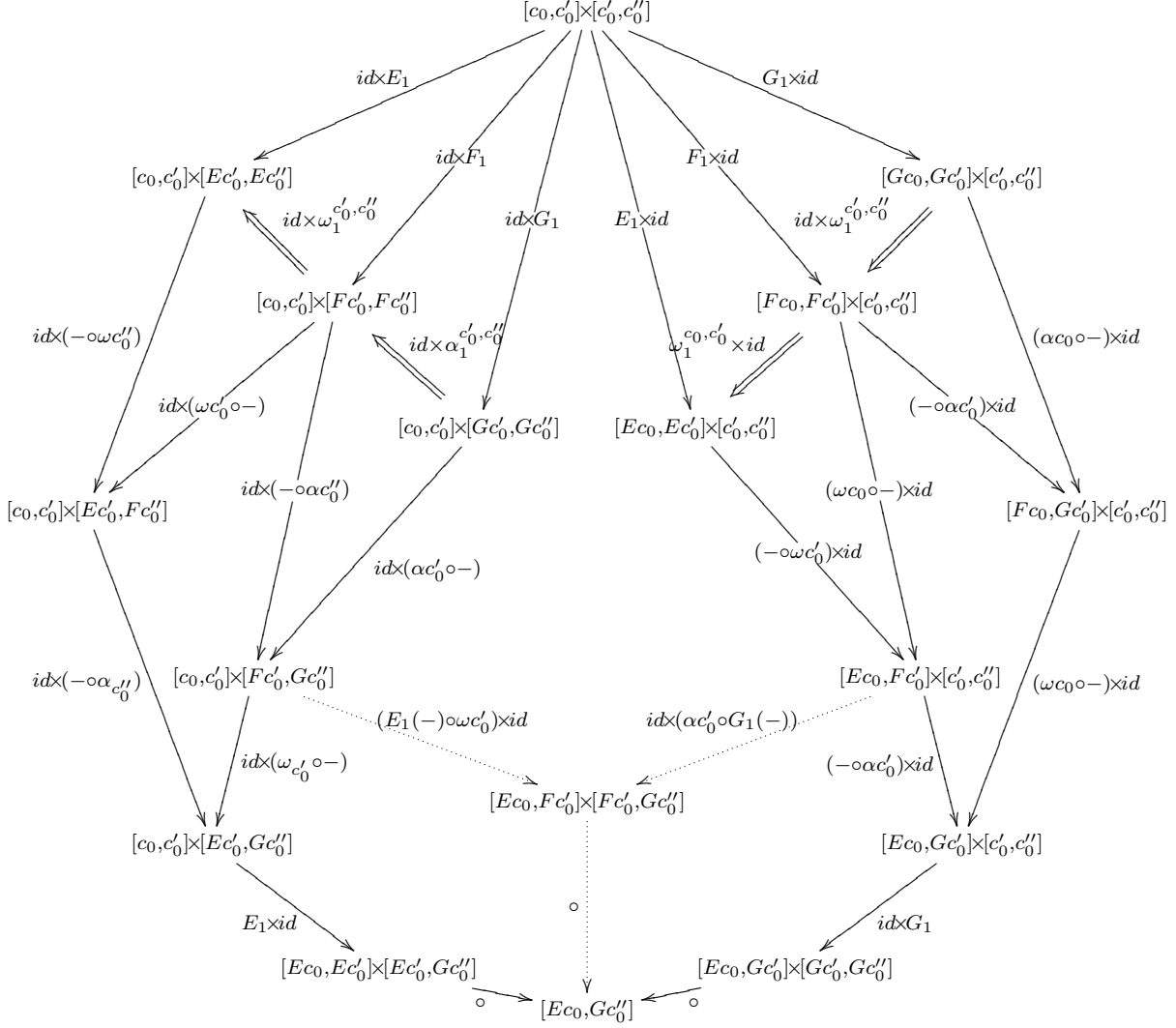
$\begin{array}{ccc} \downarrow - \circ \omega c_0 & & \downarrow \alpha c_0 \circ - \\ \downarrow - \circ \alpha c_0 & & \downarrow \omega c_0 \circ - \end{array}$

$$\begin{array}{ccc}
& \mathbb{I} & \\
& \downarrow u(c_0) & \\
\omega c_0 \swarrow & [Fc_0, Fc_0] & \searrow \alpha c_0 \\
& \downarrow id & \\
\omega c_0 \circ - \swarrow & [Ec_0, Fc_0] & \searrow - \circ \alpha c_0 \\
& \downarrow - \circ \alpha c_0 & \\
& [Ec_0, Gc_0] &
\end{array}
\quad \equiv \quad
\begin{array}{ccc}
& \mathbb{I} & \\
& \downarrow & \\
[\omega \alpha] c_0 \swarrow & [\omega \alpha] c_0 & \searrow \\
& \downarrow id & \\
[\omega \alpha] c_0 \swarrow & [\omega \alpha] c_0 & \searrow \\
& \downarrow & \\
& [Ec_0, Gc_0] &
\end{array}$$

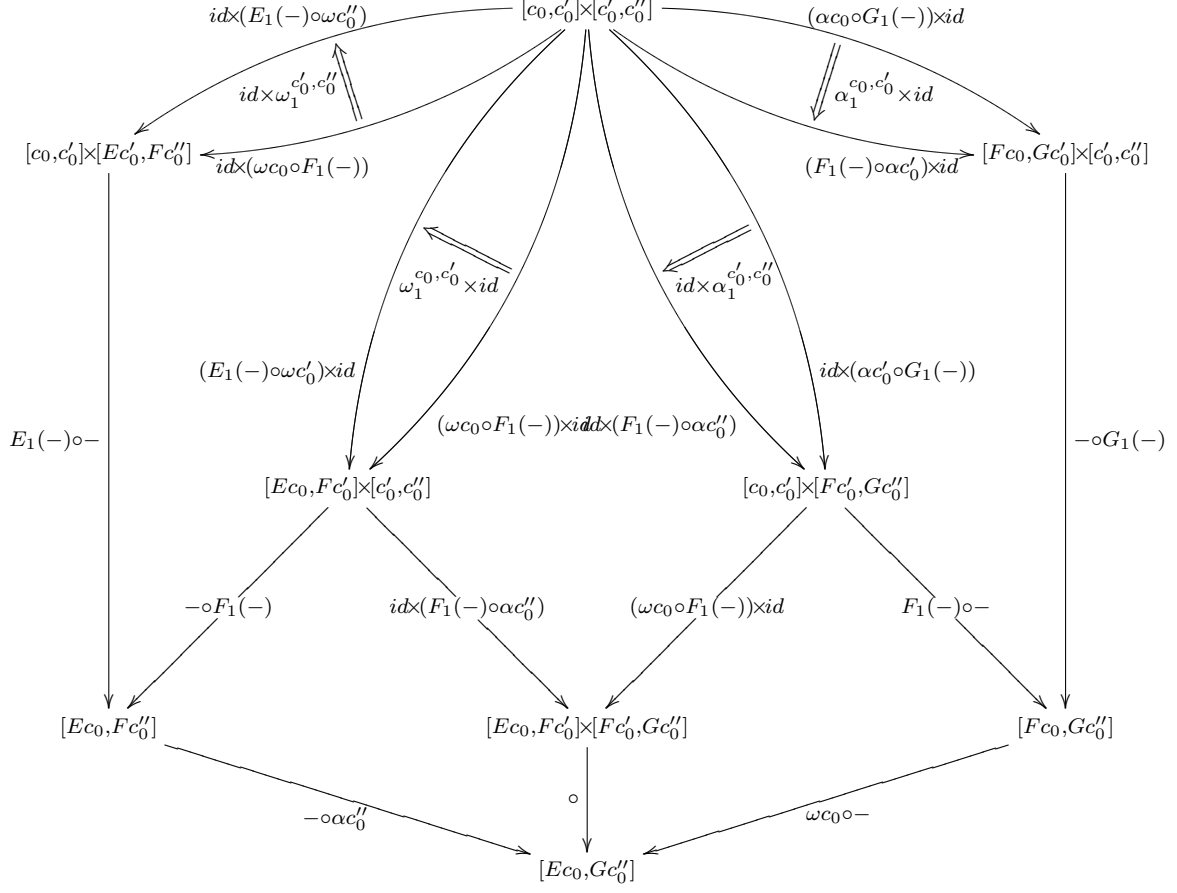
(ii)

where (i) follows for units functoriality of ω and α , while (ii) from functoriality of constant functors. This proves unit functoriality.

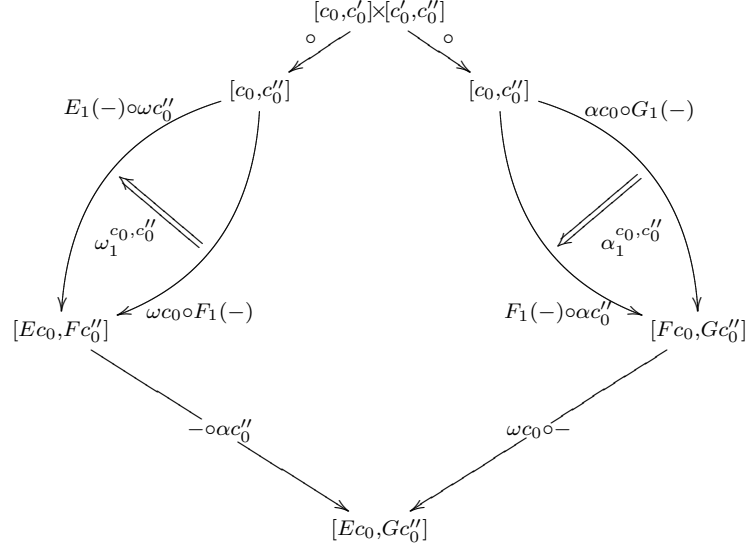
Concerning composition functoriality, take three objects c_0 , c'_0 and c''_0 in \mathbb{C} , and consider the following diagram:



After applying the product interchange (see section 2.6) to 2-morphisms $\omega_1^{c_0, c'_0}$ and $\alpha_1^{c'_0, c''_0}$, the last diagram becomes



By functoriality of 2-morphisms in $(n-1)\mathbf{Cat}$ the two sides of the diagram get



that is exactly $[\omega \bullet^1 \alpha]_1^{c_0, c''_0}$, and this concludes the proof.

3.3.2 Units

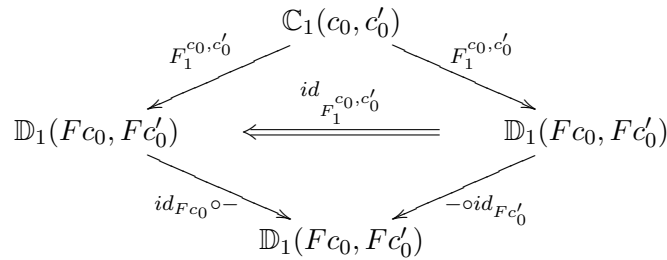
Given a morphism of n -categories $F : \mathbb{C} \xrightarrow{F} \mathbb{D}$, it is possible to define the *unit 2-cell of F* , This is denoted id_F , with

$$[id_F]_0(c_0) = id_{Fc_0} : Fc_0 \longrightarrow Fc_0$$

and

$$[id_F]_1^{c_0, c'_0} = id_{F_1^{c_0, c'_0}}$$

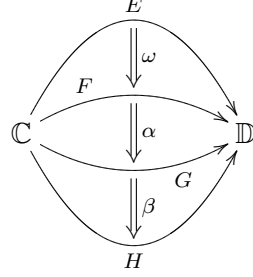
since in the diagram



$id_{Fc_0} \circ -$ and $- \circ id_{Fc'_0}$ are both equal to the identity $n - 1$ -functor over $\mathbb{D}_1(Fc_0, Fc'_0)$. It is straightforward to see that these give a 2-morphism, according to our definition.

Proposition 3.4. *Let us fix n -categories \mathbb{C} and \mathbb{D} . Morphisms between them and 2-morphisms between those form a category, with composition and units given above.*

Proof. We will sometimes denote 1-compositions of 2-morphisms just by juxtaposition. We must prove that composition is associative and units are neutral. To this end, we start considering a diagram:

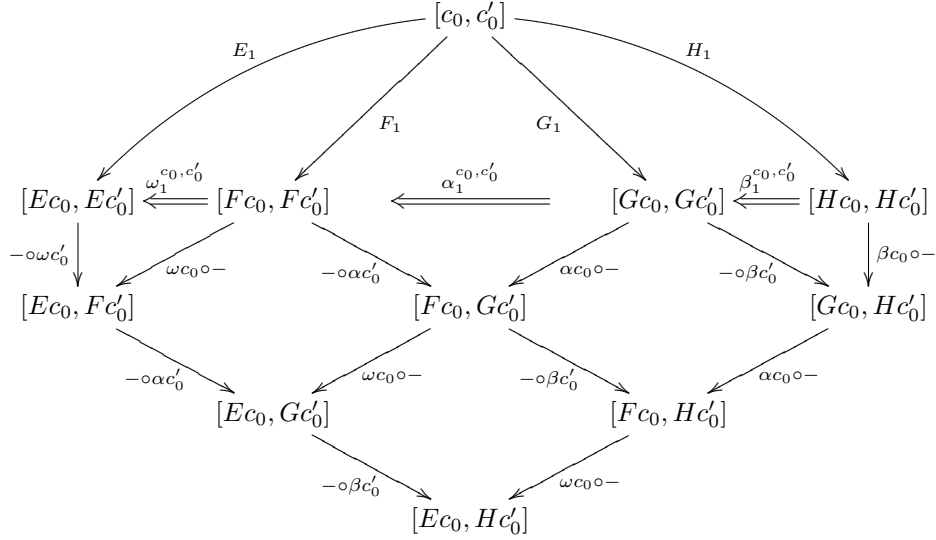


We want to prove $(\omega\alpha)\beta = \omega(\alpha\beta)$.

For every c_0 in \mathbb{C}_0 , by associative composition of maps

$$[(\omega\alpha)\beta]_0(c_0) = (\omega_0 c_0 \circ \alpha_0 c_0) \circ \beta_0 c_0 = \omega_0 c_0 \circ (\alpha_0 c_0 \circ \beta_0 c_0) = [\omega(\alpha\beta)]_0(c_0).$$

Furthermore for every pair c_0, c'_0 in \mathbb{C}_0 , associative vertical composition of 2-morphisms of $(n-1)$ -categories gives the following diagram for both $[(\omega\alpha)\beta]_1^{c_0 c'_0}$ and $[\omega(\alpha\beta)]_1^{c_0 c'_0}$



Finally, we will show that $\alpha id_G = \alpha$ ($id_F \alpha = \alpha$ is proved similarly).

For every c_0 in \mathbb{C}_0 , by neutral identities of maps

$$[\alpha id_G]_0(c_0) = \alpha_0 c_0 id_{Gc_0} = \alpha_0 c_0;$$

furthermore, for every pair c_0, c'_0 in \mathbb{C}_0 , neutral identities for vertical composition of 2-morphisms of $(n-1)$ -categories give:

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & [c_0, c'_0] & & \\
 & \swarrow F_1 & \downarrow G_1 & \searrow G_1 & \\
 [Fc_0, Fc'_0] & & & & [Gc_0, Gc'_0] \\
 \downarrow -\circ \alpha c'_0 & \swarrow \alpha_1^{c_0, c'_0} & \downarrow G_1 & \searrow id & \\
 [Fc_0, Gc'_0] & & [Gc_0, Gc'_0] & & [Gc_0, Gc'_0] \\
 \downarrow id & \swarrow \alpha c_0 \circ - & \downarrow id & \searrow id & \\
 [Fc_0, Gc'_0] & & [Gc_0, Gc'_0] & & [Gc_0, Gc'_0] \\
 \downarrow id & \swarrow \alpha c_0 \circ - & \downarrow id & \searrow \alpha c_0 \circ - & \\
 [Fc_0, Gc'_0] & & [Fc_0, Gc'_0] & & [Fc_0, Gc'_0]
 \end{array}
 \end{array}
 = \alpha_1^{c_0, c'_0}$$

□

3.4 $n\mathbf{Cat}$: the sesqui-categorical structure

In the next sections we will introduce reduced left/right compositions of morphisms and 2-morphisms of n -categories, in order to show that $n\mathbf{Cat}$ has a canonical sesqui-categorical structure. Notice that $0\mathbf{Cat} = \mathbf{Set}$ has a trivial sesqui-categorical structure (all 2-cells are identities), while $1\mathbf{Cat} = \mathbf{Cat}$ has a canonical 2-categorical structure, that inherits a sesqui-categorical structure, forgetting horizontal composition of 2-cells. Hence we may well suppose $n > 1$.

3.4.1 Defining reduced left-composition

Given the situation

$$\mathbb{B} \xrightarrow{N} \mathbb{C} \begin{array}{c} \xrightarrow{F} \mathbb{D} \\ \parallel \alpha \\ \xrightarrow{G} \mathbb{D} \end{array}$$

one defines reduced horizontal composition $N \bullet^0 \alpha : NF \Rightarrow NG : \mathbb{B} \rightarrow \mathbb{D}$ (or 0-composition) in the following way:

- for every object b_0 in \mathbb{B} ,

$$[N \bullet^0 \alpha]_0 = \alpha_0(N(b_0)) : F(N(b_0)) \rightarrow G(N(b_0))$$

• for every pair of objects b_0, b'_0 of \mathbb{B} , the diagram below describes $[N \bullet^0 \alpha]_1^{b_0, b'_0}$ by means of reduced left composition in $(n-1)\mathbf{Cat}$:

$$[N \bullet^0 \alpha]_1^{b_0, b'_0} = N_1^{b_0, b'_0} \bullet^0 \alpha_1^{Nb_0, Nb'_0}$$

$$\begin{array}{c}
 \mathbb{B}_1(b_0, b'_0) \\
 \downarrow N_1 \\
 \begin{array}{ccc}
 [NF]_1 & \mathbb{C}_1(Nb_0, Nb'_0) & [NG]_1 \\
 \swarrow F_1 & & \searrow G_1 \\
 \mathbb{D}_1(NF(b_0), NF(b'_0)) & \xleftarrow{\alpha_1^{Nb_0, Nb'_0}} & \mathbb{D}_1(NG(b_0), NG(b'_0)) \\
 \swarrow -\circ \alpha_{Nb'_0} & & \swarrow \alpha_{Nb_0} \circ - \\
 & \mathbb{D}_1(NF(b_0), NG(b'_0)) &
 \end{array}
 \end{array}$$

To prove that these data give indeed a 2-morphism, unit and composition axioms equations (3.6) (3.5) must hold.

To this purpose, let us chose first an object b_0 of \mathbb{B} and consider the following chain of diagrams equalities:

$$\begin{array}{ccc}
 \begin{array}{c}
 \mathbb{I} \\
 \downarrow u(b_0) \\
 [b_0, b_0] \\
 \begin{array}{ccc}
 [NF]_1 & \downarrow N_1 & [NG]_1 \\
 \swarrow F_1 & [Nb_0, Nb_0] & \searrow G_1 \\
 [NF(b_0), NF(b_0)] & \xleftarrow{\alpha_1^{Nb_0, Nb_0}} & [NG(b_0), NG(b_0)] \\
 \swarrow -\circ \alpha_{Nb_0} & & \swarrow \alpha_{Nb_0} \circ - \\
 & [NF(b_0), NG(b_0)] &
 \end{array}
 \end{array}
 & \stackrel{(i)}{=} &
 \begin{array}{c}
 \mathbb{I} \\
 \downarrow u(Nb_0) \\
 [Nb_0, Nb_0] \\
 \begin{array}{ccc}
 F_1 & & G_1 \\
 \swarrow & [NF(b_0), NF(b_0)] \xleftarrow{\alpha_1^{Nb_0, Nb_0}} [NG(b_0), NG(b_0)] & \searrow \\
 \swarrow -\circ \alpha_{Nb_0} & & \swarrow \alpha_{Nb_0} \circ - \\
 & [NF(b_0), NG(b_0)] &
 \end{array}
 \end{array}
 & \stackrel{(ii)}{=} &
 \begin{array}{c}
 \mathbb{I} \\
 \downarrow id \\
 [NF(b_0), NG(b_0)]
 \end{array}
 \end{array}$$

(i) holds by functoriality w.r.t. units (3.4), and (ii) is simply functoriality w.r.t units of α (3.6) for the object Nb_0 of \mathbb{C} .

Concerning composition axiom, take three objects b_0, b'_0 and b''_0 in \mathbb{B} , and consider the following diagram:

$$\begin{array}{c}
\begin{array}{ccc}
& [b_0, b'_0] \times [b'_0, b''_0] & \\
\swarrow \scriptstyle{id \times N_1} & & \searrow \scriptstyle{N_1 \times id} \\
[b_0, b'_0] \times [Nb'_0, Nb''_0] & & [Nb_0, Nb'_0] \times [b'_0, b''_0] \\
\swarrow \scriptstyle{id \times F_1} & \downarrow \scriptstyle{id \times G_1} & \searrow \scriptstyle{G_1 \times id} \\
[b_0, b'_0] \times [NF(b'_0), NF(b''_0)] & & [NG(b_0), NG(b'_0)] \times [b'_0, b''_0] \\
\downarrow \scriptstyle{id \times (-\circ \alpha_{Nb'_0})} & \swarrow \scriptstyle{id \times \alpha_1^{Nb'_0, Nb''_0}} & \downarrow \scriptstyle{F_1 \times id} \\
[b_0, b'_0] \times [NF(b'_0), NG(b''_0)] & [b_0, b'_0] \times [NG(b'_0), NG(b''_0)] & [NF(b_0), NF(b'_0)] \times [b'_0, b''_0] \\
\downarrow \scriptstyle{N_1 \times id} & \swarrow \scriptstyle{id \times (\alpha_{Nb'_0} \circ -)} & \searrow \scriptstyle{(-\circ \alpha_{Nb'_0}) \times id} \\
[Nb_0, Nb'_0] \times [NF(b'_0), NG(b''_0)] & & [NF(b_0), NG(b'_0)] \times [b'_0, b''_0] \\
\downarrow \scriptstyle{F_1 \times id} & & \downarrow \scriptstyle{id \times N_1} \\
[NF(b_0), NF(b'_0)] \times [NF(b'_0), NG(b''_0)] & & [NF(b_0), NG(b'_0)] \times [Nb'_0, Nb''_0] \\
& \searrow \scriptstyle{\circ} & \downarrow \scriptstyle{id \times G_1} \\
& [NF(b_0), NG(b''_0)] & [NF(b_0), NG(b'_0)] \times [NG(b'_0), NG(b''_0)] \\
& \swarrow \scriptstyle{\circ} & \\
& [NF(b_0), NG(b''_0)] &
\end{array}
\end{array}$$

By product interchange rules (see *Lemma 2.11*, when one of the components is an identity) 2-cells $id \times \alpha_1$ and $\alpha_1 \times id$ can slide along $N_1 \times id$ and $id \times N_1$ respectively, in order to give:

$$\begin{array}{c}
\begin{array}{c}
[b_0, b'_0] \times [b'_0, b''_0] \\
\swarrow N_1 \times N_1 \quad \searrow N_1 \times N_1 \\
[Nb_0, Nb'_0] \times [Nb'_0, Nb''_0] \quad \equiv \quad [Nb_0, Nb'_0] \times [Nb'_0, Nb''_0] \\
\swarrow id \times F_1 \quad \downarrow id \times G_1 \quad \searrow G_1 \times id \\
[Nb_0, Nb'_0] \times [NF(b'_0), NF(b''_0)] \quad [Nb_0, Nb'_0] \times [NG(b'_0), NG(b''_0)] \quad [NF(b_0), NF(b'_0)] \times [Nb'_0, Nb''_0] \\
\swarrow id \times (-\circ \alpha_{Nb'_0}) \quad \swarrow id \times \alpha_1^{Nb'_0, Nb''_0} \quad \swarrow F_1 \times id \quad \swarrow (-\circ \alpha_{Nb'_0}) \times id \quad \swarrow (\alpha_{Nb_0}) \times id \\
[Nb_0, Nb'_0] \times [NF(b'_0), NG(b''_0)] \quad [NF(b_0), NF(b'_0)] \times [Nb'_0, Nb''_0] \quad [NF(b_0), NG(b'_0)] \times [Nb'_0, Nb''_0] \\
\downarrow F_1 \times id \quad \swarrow id \times (\alpha_{Nb'_0} \circ -) \quad \swarrow (-\circ \alpha_{Nb'_0}) \times id \quad \downarrow id \times G_1 \\
[NF(b_0), NF(b'_0)] \times [NF(b'_0), NG(b''_0)] \quad [NF(b_0), NG(b'_0)] \times [NG(b'_0), NG(b''_0)] \\
\searrow \circ \quad \swarrow \circ \\
[NF(b_0), NG(b''_0)]
\end{array}
\end{array}$$

now, just apply composition functoriality for α (3.5) and get:

$$\begin{array}{c}
\begin{array}{c}
[b_0, b'_0] \times [b'_0, b''_0] \\
\downarrow N_1 \times N_1 \\
[Nb_0, Nb'_0] \times [Nb'_0, Nb''_0] \\
\downarrow \circ \\
[Nb_0, Nb''_0] \\
\swarrow F_1 \quad \searrow G_1 \\
[NF(b_0), NF(b''_0)] \quad [NG(b_0), NG(b''_0)] \\
\swarrow \alpha_1^{Nb_0, Nb''_0} \\
[NF(b_0), NG(b''_0)] \\
\swarrow -\circ \alpha_{Nb'_0} \quad \swarrow \alpha_{Nb_0} \circ - \\
[NF(b_0), NG(b''_0)]
\end{array}
\quad = \quad
\begin{array}{c}
[b_0, b'_0] \times [b'_0, b''_0] \\
\downarrow \circ \\
[Nb_0, Nb''_0] \\
\downarrow N_1 \\
[Nb_0, Nb''_0] \\
\swarrow F_1 \quad \searrow G_1 \\
[NF(b_0), NF(b''_0)] \quad [NG(b_0), NG(b''_0)] \\
\swarrow \alpha_1^{Nb_0, Nb''_0} \\
[NF(b_0), NG(b''_0)] \\
\swarrow -\circ \alpha_{Nb'_0} \quad \swarrow \alpha_{Nb_0} \circ - \\
[NF(b_0), NG(b''_0)]
\end{array}
\end{array}$$

where the last equality is functoriality of N w.r.t. compositions (3.3). And this completes the proof that left horizontal composition is well defined.

3.4.2 Left-composition axioms

Given the situation

$$\begin{array}{ccccc} \mathbb{A} & \xrightarrow{M} & \mathbb{B} & \xrightarrow{N} & \mathbb{C} \\ & & & & \downarrow \alpha \\ & & & & \downarrow \beta \\ & & & & \downarrow \gamma \\ & & & & \downarrow \delta \\ & & & & \downarrow \epsilon \\ & & & & \downarrow \zeta \\ & & & & \downarrow \eta \\ & & & & \downarrow \theta \\ & & & & \downarrow \iota \\ & & & & \downarrow \kappa \\ & & & & \downarrow \lambda \\ & & & & \downarrow \mu \\ & & & & \downarrow \nu \\ & & & & \downarrow \xi \\ & & & & \downarrow \omicron \\ & & & & \downarrow \pi \\ & & & & \downarrow \rho \\ & & & & \downarrow \sigma \\ & & & & \downarrow \tau \\ & & & & \downarrow \upsilon \\ & & & & \downarrow \phi \\ & & & & \downarrow \chi \\ & & & & \downarrow \psi \\ & & & & \downarrow \omega \\ & & & & \downarrow \delta \\ & & & & \downarrow \epsilon \\ & & & & \downarrow \zeta \\ & & & & \downarrow \eta \\ & & & & \downarrow \theta \\ & & & & \downarrow \iota \\ & & & & \downarrow \kappa \\ & & & & \downarrow \lambda \\ & & & & \downarrow \mu \\ & & & & \downarrow \nu \\ & & & & \downarrow \xi \\ & & & & \downarrow \omicron \\ & & & & \downarrow \pi \\ & & & & \downarrow \rho \\ & & & & \downarrow \sigma \\ & & & & \downarrow \tau \\ & & & & \downarrow \upsilon \\ & & & & \downarrow \phi \\ & & & & \downarrow \chi \\ & & & & \downarrow \psi \\ & & & & \downarrow \omega \end{array}$$

in $n\mathbf{Cat}$, left-composition defined above satisfies axioms (L1) to (L4) of *Proposition 2.2*.

(L1)

$$Id_{\mathbb{C}} \bullet^0 \alpha = \alpha$$

Proof. Let objects c_0, c'_0 of \mathbb{C} be given. It is clear that

$$[Id_{\mathbb{C}} \bullet^0 \alpha]_{c_0} \stackrel{(def)}{=} \alpha_{Id_{\mathbb{C}}(c_0)} = \alpha_{c_0}$$

and also that

$$[Id_{\mathbb{C}} \bullet^0 \alpha]_1^{c_0, c'_0} \stackrel{(def)}{=} [Id_{\mathbb{C}}]_1^{c_0, c'_0} \bullet^0 \alpha_1^{c_0, c'_0} \stackrel{(1)}{=} Id_{\mathbb{C}_1(c_0, c'_0)} \bullet^0 \alpha_1^{c_0, c'_0} \stackrel{(2)}{=} \alpha_1^{c_0, c'_0}$$

where (1) comes from the definition of *identity functors*, and (2) is axiom (L1) for $(n-1)\mathbf{Cat}$. □

(L2)

$$MN \bullet^0 \alpha = M \bullet^0 (N \bullet^0 \alpha)$$

Proof. Let objects a_0, a'_0 of \mathbb{A} be given. Then

$$[MN \bullet^0 \alpha]_{a_0} \stackrel{(def)}{=} \alpha_{MN(a_0)} = \alpha_{N(Ma_0)} \stackrel{(def)}{=} [N \bullet^0 \alpha]_{Ma_0} \stackrel{(def)}{=} [M \bullet^0 (N \bullet^0 \alpha)]_{a_0}$$

Furthermore,

$$\begin{aligned} [MN \bullet^0 \alpha]_1^{a_0, a'_0} &\stackrel{(def)}{=} [MN]_1^{a_0, a'_0} \bullet^0 \alpha_1^{MN(a_0), MN(a'_0)} \\ &= M_1^{a_0, a'_0} N_1^{Ma_0, Ma'_0} \bullet^0 \alpha_1^{MN(a_0), MN(a'_0)} \\ &\stackrel{(1)}{=} M_1^{a_0, a'_0} \bullet^0 (N_1^{Ma_0, Ma'_0} \bullet^0 \alpha_1^{N(Ma_0), N(Ma'_0)}) \\ &\stackrel{(def)}{=} M_1^{a_0, a'_0} \bullet^0 [N \bullet^0 \alpha]_1^{Ma_0, Ma'_0} \\ &\stackrel{(def)}{=} [M \bullet^0 (N \bullet^0 \alpha)]_1^{a_0, a'_0} \end{aligned}$$

where (1) is axiom (L2) for $(n-1)\mathbf{Cat}$. □

(L3)

$$N \bullet^0 id_F = id_{NF}$$

Proof. Let objects b_0, b'_0 of \mathbb{B} be given. Trivially,

$$[N \bullet^0 id_F]_{b_0} \stackrel{(def)}{=} [id_F]_{Nb_0} = [id_{NF}]_{b_0}$$

and

$$\begin{aligned} [N \bullet^0 id_F]_1^{b_0, b'_0} &\stackrel{(def)}{=} N_1^{b_0, b'_0} \bullet^0 [id_F]_1^{Nb_0, Nb'_0} \stackrel{(1)}{=} N_1^{b_0, b'_0} \bullet^0 id_{F_1^{Nb_0, Nb'_0}} = \\ &\stackrel{(2)}{=} id_{N_1^{b_0, b'_0} F_1^{Nb_0, Nb'_0}} = id_{[NF]_1^{b_0, b'_0}} \stackrel{(def)}{=} [id_{NF}]_1^{b_0, b'_0} \end{aligned}$$

where (1) comes from the definition of identity transformation and (2) is axiom (L3) in $(n-1)\mathbf{Cat}$. \square

(L4)

$$N \bullet^0 (\alpha \bullet^1 \beta) = (N \bullet^0 \alpha) \bullet^1 (N \bullet^0 \beta)$$

Proof. Let objects b_0, b'_0 of \mathbb{B} be given. On objects:

$$[N \bullet^0 (\alpha \bullet^1 \beta)]_{b_0} \stackrel{(def)}{=} [\alpha \bullet^1 \beta]_{Nb_0} = \alpha_{Nb_0} \circ \beta_{Nb_0} \stackrel{(def)}{=} [N \bullet^0 \alpha]_{b_0} \circ [N \bullet^0 \beta]_{b_0} = [(N \bullet^0 \alpha) \bullet^1 (N \bullet^0 \beta)]_{b_0}$$

On homs:

$$\begin{aligned} [N \bullet^0 (\alpha \bullet^1 \beta)]_1^{b_0, b'_0} &\stackrel{(def)}{=} N_1^{b_0, b'_0} \bullet^0 [\alpha \bullet^1 \beta]_1^{Nb_0, Nb'_0} \\ &\stackrel{(1)}{=} N_1^{b_0, b'_0} \bullet^0 \left((\beta_1^{Nb_0, Nb'_0} \bullet^0 (\alpha_{Nb_0} \circ -)) \bullet^1 (\alpha_1^{Nb_0, Nb'_0} \bullet^0 (- \circ \beta_1^{Nb_0, Nb'_0})) \right) \\ &\stackrel{(2)}{=} (N_1^{b_0, b'_0} \bullet^0 \beta_1^{Nb_0, Nb'_0} \bullet^0 (\alpha_{Nb_0} \circ -)) \bullet^1 (N_1^{b_0, b'_0} \bullet^0 \alpha_1^{Nb_0, Nb'_0} \bullet^0 (- \circ \beta_1^{Nb_0, Nb'_0})) \\ &\stackrel{(def)}{=} ([N \bullet^0 \beta]_1^{b_0, b'_0} \bullet^0 ([N \bullet^0 \alpha]_{b_0} \circ -)) \bullet^1 ([N \bullet^0 \alpha]_1^{b_0, b'_0} \bullet^0 (- \circ [N \bullet^0 \beta]_{b'_0})) \\ &\stackrel{(3)}{=} [(N \bullet^0 \alpha) \bullet^1 (N \bullet^0 \beta)]_1^{b_0, b'_0} \end{aligned}$$

where (1) and (3) hold by definition of vertical composites of 2-morphisms, (2) by axiom (L4) in $(n-1)\mathbf{Cat}$. \square

3.4.3 Defining reduced right-composition

Given the situation

$$\begin{array}{ccccc} & & F & & \\ & \curvearrowright & \parallel & \curvearrowright & \\ \mathbb{C} & & \alpha & & \mathbb{D} \xrightarrow{L} \mathbb{E} \\ & \curvearrowleft & \downarrow & \curvearrowleft & \\ & & G & & \end{array}$$

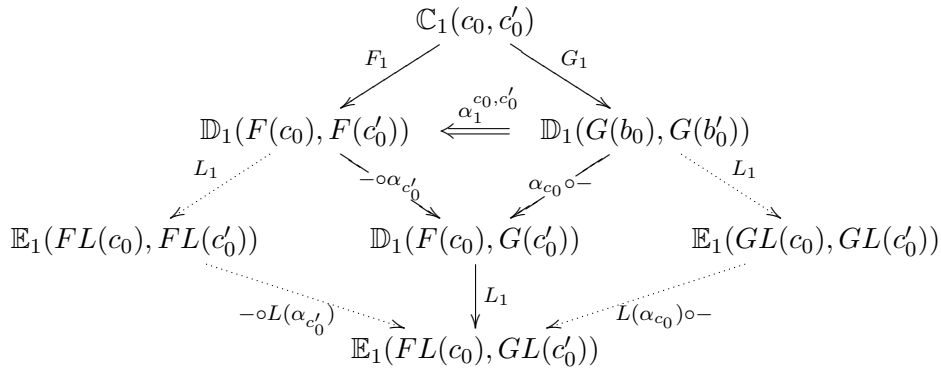
one defines reduced horizontal composition $\alpha \bullet^0 L : FL \Rightarrow GL : \mathbb{C} \rightarrow \mathbb{E}$ (or 0-composition) in the following way:

- for every object c_0 in \mathbb{C} ,

$$[\alpha \bullet^0 L]_0 = L(\alpha_0(c_0)) : L(F(c_0)) \rightarrow L(G(c_0))$$

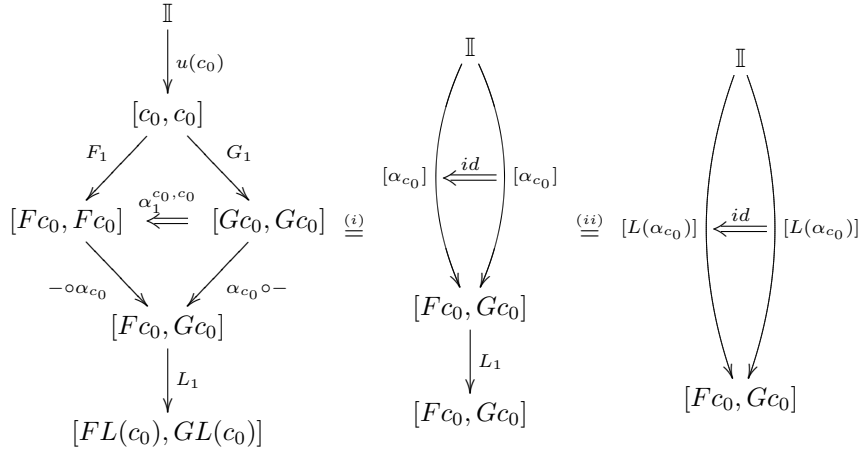
- for every pair of objects c_0, c'_0 of \mathbb{B} , the diagram below describes $[\alpha \bullet^0 L]_1^{c_0, c'_0}$ by means of reduced right composition in $(n-1)\mathbf{Cat}$:

$$[\alpha \bullet^0 L]_1^{c_0, c'_0} = \alpha_1^{c_0, c'_0} \bullet^0 L_1^{F c_0, G c'_0}$$



To prove that these data give indeed a 2-morphism, unit and composition axioms equations (3.6) (3.5) must hold.

To this end, chose first an object c_0 of \mathbb{C} and consider the following chain of diagrams equalities:



where (i) holds by unit functoriality of α (3.6), (ii) by functoriality w.r.t. units (3.4) and by axiom (R3) of reduced right composition in the sesqui-category $(n-1)\mathbf{Cat}$.

Concerning composition axiom, let us take three objects c_0, c'_0 and c''_0 in \mathbb{C} , and consider the following diagram:

$$\begin{array}{ccccc}
 & & [c_0, c'_0] \times [c'_0, c''_0] & & \\
 & \swarrow^{id \times (F(-) \circ \alpha_{c'_0}''')} & & \searrow^{((\alpha_{c_0} \circ G(-)) \times id)} & \\
 [c_0, c'_0] \times [Fc'_0, Gc''_0] & & & & [Fc_0, Gc'_0] \times [c'_0, c''_0] \\
 \uparrow^{id \times \alpha_1^{c'_0, c''_0}} & & & & \downarrow^{\alpha_1^{c_0, c'_0} \times id} \\
 & \swarrow^{id \times (\alpha_{c'_0} \circ G(-))} & & \searrow^{(F(-) \circ \alpha_{c'_0}) \times id} & \\
 & [Fc_0, Fc'_0] \times [Fc'_0, Gc''_0] & & [Fc_0, Gc'_0] \times [Gc'_0, Gc''_0] & \\
 \downarrow^{id \times L_1} & \swarrow^{F_1 \times id} & & \swarrow^{id \times G_1} & \downarrow^{L_1 \times id} \\
 [c_0, c'_0] \times [FL(c'_0), GL(c''_0)] & & [Fc_0, Gc''_0] & & [FL(c_0), GL(c'_0)] \times [c'_0, c''_0] \\
 \downarrow^{[FL]_1 \times id} & \swarrow^{\circ} & \downarrow^{L_1} & \swarrow^{\circ} & \downarrow^{id \times [GL]_1} \\
 [FL(c_0), FL(c'_0)] \times [FL(c'_0), GL(c''_0)] & & [FL(c_0), GL(c''_0)] & & [FL(c_0), GL(c'_0)] \times [GL(c'_0), GL(c''_0)] \\
 & \swarrow^{\circ} & & \swarrow^{\circ} & \\
 & [FL(c_0), GL(c''_0)] & & &
 \end{array}$$

Here, internal dotted construction commutes with external (by product properties), hence it can take its place and suggests to apply composition functoriality (3.5) for α , in order to give

$$\begin{array}{ccccc}
 & & [c_0, c'_0] \times [c'_0, c''_0] & & \\
 & & \downarrow^{\circ} & & \\
 & & [c_0, c''_0] & & \\
 & \swarrow^{F_1} & & \searrow^{G_1} & \\
 [Fc_0, Fc_0] & & [c_0, c''_0] & & [Gc_0, Gc_0] \\
 & \swarrow^{\alpha_1^{c_0, c''_0}} & & \swarrow^{\alpha_1^{c_0, c''_0}} & \\
 & [Fc_0, Gc''_0] & & [Gc_0, Gc''_0] & \\
 & \swarrow^{-\circ \alpha_{c'_0}'''} & & \swarrow^{\alpha_{c_0} \circ -} & \\
 & [Fc_0, Gc''_0] & & [Gc_0, Gc''_0] & \\
 & \downarrow^{L_1} & & & \\
 & [FL(c_0), GL(c''_0)] & & &
 \end{array}$$

and this complete the proof horizontal right composition is well defined.

3.4.4 Right-composition axioms

Given the diagram

$$\begin{array}{ccccc}
 & & F & & \\
 & \nearrow & \Downarrow \alpha & \searrow & \\
 \mathbb{C} & \xrightarrow{G} & \mathbb{D} & \xrightarrow{L} & \mathbb{E} \xrightarrow{M} \mathbb{F} \\
 & \searrow & \Downarrow \beta & \nearrow & \\
 & & H & &
 \end{array}$$

in $n\mathbf{Cat}$, right-composition defined above satisfies axioms (R1) to (R4) of *Proposition 2.2*.

(R1)

$$\alpha \circ Id_{\mathbb{D}} = \alpha$$

Proof. Let objects c_0, c'_0 of \mathbb{C} be given. It is clear that

$$[\alpha \bullet^0 Id_{\mathbb{D}}]_{c_0} \stackrel{(def)}{=} Id_{\mathbb{D}}(\alpha_{c_0}) = \alpha_{c_0}$$

and also that

$$[\alpha \bullet^0 Id_{\mathbb{D}}]_1^{c_0, c'_0} \stackrel{(def)}{=} \alpha_1^{c_0, c'_0} \bullet^0 [Id_{\mathbb{D}}]_1^{F c_0, G c'_0} \stackrel{(1)}{=} \alpha_1^{c_0, c'_0} \bullet^0 Id_{\mathbb{D}_1(F c_0, G c'_0)} \stackrel{(2)}{=} \alpha_1^{c_0, c'_0}$$

where (1) comes from the definition of *identity functors*, and (2) is axiom (R1) for $(n-1)\mathbf{Cat}$. □

(R2)

$$\alpha \bullet^0 LM = (\alpha \bullet^0 L) \bullet^0 M$$

Proof. Let objects c_0, c'_0 of \mathbb{C} be given. Then

$$[\alpha \bullet^0 LM]_{c_0} \stackrel{(def)}{=} LM(\alpha_{c_0}) = M(L(\alpha_{c_0})) \stackrel{(def)}{=} M([\alpha \bullet^0 L]_{c_0}) \stackrel{(def)}{=} [(\alpha \bullet^0 L) \bullet^0 M]_{c_0}$$

Furthermore

$$\begin{aligned}
 [\alpha \bullet^0 LM]_1^{c_0, c'_0} & \stackrel{(def)}{=} \alpha_1^{c_0, c'_0} \bullet^0 [LM]_1^{F c_0, G c'_0} \\
 & = \alpha_1^{c_0, c'_0} \bullet^0 (L_1^{F c_0, G c'_0} M_1^{L(F c_0), L(G c'_0)}) \\
 & \stackrel{(1)}{=} (\alpha_1^{c_0, c'_0} \bullet^0 L_1^{F c_0, G c'_0}) \bullet^0 M_1^{L(F c_0), L(G c'_0)} \\
 & \stackrel{(def)}{=} [\alpha \bullet^0 L]_1^{c_0, c'_0} \bullet^0 M_1^{L(F c_0), L(G c'_0)} \\
 & \stackrel{(def)}{=} [(\alpha \bullet^0 L) \bullet^0 M]_1^{c_0, c'_0}
 \end{aligned}$$

where (1) is axiom (R2) for $(n-1)\mathbf{Cat}$. □

(L3)

$$id_F \bullet^0 L = id_{FL}$$

Proof. Let objects c_0, c'_0 of \mathbb{C} be given. Trivially,

$$[id_F \bullet^0 L]_{c_0} \stackrel{(def)}{=} L([id_F]_{c_0}) \stackrel{(1)}{=} L(id_{Fc_0}) \stackrel{(2)}{=} id_{FL(c_0)}$$

where (1) holds by definition of identity transformations and (2) from functoriality of L . Furthermore,

$$\begin{aligned} [id_F \bullet^0 L]_1^{c_0, c'_0} &\stackrel{(def)}{=} [id_F]_1^{c_0, c'_0} \bullet^0 L_1^{Fc_0, Fc'_0} \stackrel{(1)}{=} id_{F_1^{c_0, c'_0}} \bullet^0 L_1^{Fc_0, Fc'_0} = \\ &\stackrel{(2)}{=} id_{F_1^{c_0, c'_0} L_1^{Fc_0, Fc'_0}} = id_{[FL]_1^{c_0, c'_0}} \stackrel{(def)}{=} [id_{FL}]_1^{c_0, c'_0} \end{aligned}$$

where (1) comes from the definition of identity transformation and (2) is axiom (R3) in $(n-1)\mathbf{Cat}$. \square

(L4)

$$(\alpha \bullet^1 \beta) \bullet^0 L = (\alpha \bullet^0 L) \bullet^1 (\beta \bullet^0 L)$$

Proof. Let objects c_0, c'_0 of \mathbb{C} be given. On objects:

$$[(\alpha \bullet^1 \beta) \bullet^0 L]_{c_0} \stackrel{(def)}{=} L([\alpha \bullet^1 \beta]_{c_0}) = L(\alpha_{c_0} \circ \beta_{c_0}) = L(\alpha_{c_0}) \circ L(\beta_{c_0}) = [(\alpha \bullet^0 L) \bullet^1 (\beta \bullet^0 L)]_{c_0}$$

On homs:

$$[(\alpha \bullet^1 \beta) \bullet^0 L]_1^{c_0, c'_0} =$$

$$\begin{aligned} &\stackrel{(def)}{=} [\alpha \bullet^1 \beta]_1^{c_0, c'_0} \bullet^0 L_1^{Fc_0, Hc'_0} \\ &\stackrel{(1)}{=} \left((\beta_1^{c_0, c'_0} \bullet^0 (\alpha_{c_0} \circ -)) \bullet^1 (\alpha_1^{c_0, c'_0} \bullet^0 (- \circ \beta_{c'_0})) \right) \bullet^0 L_1^{Fc_0, Hc'_0} \\ &\stackrel{(2)}{=} \left((\beta_1^{c_0, c'_0} \bullet^0 (\alpha_{c_0} \circ -)) \bullet^0 L_1^{Fc_0, Hc'_0} \right) \bullet^1 \left((\alpha_1^{c_0, c'_0} \bullet^0 (- \circ \beta_{c'_0})) \bullet^0 L_1^{Fc_0, Hc'_0} \right) \\ &\stackrel{(3)}{=} \left(\beta_1^{c_0, c'_0} \bullet^0 (L_1^{Gc_0, Hc'_0} (L(\alpha_{c_0}) \circ -)) \right) \bullet^1 \left(\alpha_1^{c_0, c'_0} \bullet^0 (L_1^{Fc_0, Gc'_0} (- \circ L(\beta_{c'_0}))) \right) \\ &\stackrel{(4)}{=} \left((\beta_1^{c_0, c'_0} \bullet^0 L_1^{Gc_0, Hc'_0}) \bullet^0 (L(\alpha_{c_0}) \circ -) \right) \bullet^1 \left((\alpha_1^{c_0, c'_0} \bullet^0 L_1^{Fc_0, Gc'_0}) \bullet^0 (- \circ L(\beta_{c'_0})) \right) \\ &\stackrel{(def)}{=} ([\beta \bullet^0 L]_1^{c_0, c'_0} \bullet^0 ([\alpha \bullet^0 L]_{c_0} \circ -)) \bullet^1 ([\alpha \bullet^0 L]_1^{c_0, c'_0} \bullet^0 (- \circ [\beta \bullet^0 L]_{c'_0})) \\ &\stackrel{(5)}{=} [(\alpha \bullet^0 L) \bullet^1 (\beta \bullet^0 L)]_1^{c_0, c'_0} \end{aligned}$$

where (1) and (5) hold by definition of vertical composites of 2-morphisms, (2) by axiom (R4) in $(n-1)\mathbf{Cat}$, (3) by functoriality of L w.r.t. 0-composition, and (4) by axiom (R2) in $(n-1)\mathbf{Cat}$. \square

3.4.5 Whiskering axiom

Given the situation

$$\begin{array}{ccccc} \mathbb{B} & \xrightarrow{N} & \mathbb{C} & \begin{array}{c} \xrightarrow{F} \\ \parallel \alpha \\ \xrightarrow{G} \end{array} & \mathbb{D} \xrightarrow{L} \mathbb{E} \end{array}$$

a whiskering operation may be defined if the following equation holds:

(LR5)

$$(N \circ \alpha) \circ L = N \circ (\alpha \circ L)$$

Proof. Let objects b_0, b'_0 of \mathbb{B} be given. Then the following follows immediately from definitions

$$[(N \bullet^0 \alpha) \bullet^0 L]_{b_0} = L([N \bullet^0 \alpha]_{b_0}) = L(\alpha_{Nb_0}) = [\alpha \bullet^0 L]_{Nb_0} = [N \circ (\alpha \circ L)]_{b_0}$$

Analogously, consider:

$$\begin{aligned} [(N \bullet^0 \alpha) \bullet^0 L]_1^{b_0, b'_0} &= [N \bullet^0 \alpha]_1^{b_0, b'_0} \bullet^0 L_1^{F(Nb_0), G(Nb'_0)} \\ &= \left(N_1^{b_0, b'_0} \bullet^0 \alpha_1^{Nb_0, Nb'_0} \right) \bullet^0 L_1^{F(Nb_0), G(Nb'_0)} \\ &\stackrel{(1)}{=} N_1^{b_0, b'_0} \bullet^0 \left(\alpha_1^{Nb_0, Nb'_0} \bullet^0 L_1^{F(Nb_0), G(Nb'_0)} \right) \\ &= N_1^{b_0, b'_0} \bullet^0 [\alpha \bullet^0 L]_1^{Nb_0, Nb'_0} \\ &= [N \bullet^0 (\alpha \bullet^0 L)]_1^{b_0, b'_0} \end{aligned}$$

where everything comes directly from definitions, but (1) that is exactly the whiskering in $(n-1)\mathbf{Cat}$. \square

3.5 Products in $n\mathbf{Cat}$

In order to close the induction on the definition of $n\mathbf{Cat}$, all we need is to show that it admits finite products, according to the 2-dimensional *Universal Property 2.9*, i.e. to show it admits binary products and terminal objects.

3.5.1 2-universality of categorical products

Let two n -categories \mathbb{C} and \mathbb{D} be given. We know from Proposition 3.3 that the underlying category $[n\mathbf{Cat}]$ admits a (standard) product of \mathbb{C} and \mathbb{D} :

$$\begin{array}{ccc} & \mathbb{C} \times \mathbb{D} & \\ \Pi_{\mathbb{C}} \swarrow & & \searrow \Pi_{\mathbb{D}} \\ \mathbb{C} & & \mathbb{D} \end{array}$$

Now suppose we are given two 2-morphisms

$$\alpha : A \Rightarrow A' : \mathbb{X} \rightarrow \mathbb{C} \times \mathbb{D}, \quad \beta : B \Rightarrow B' : \mathbb{X} \rightarrow \mathbb{C} \times \mathbb{D}$$

According to *Universal Property 2.9*, what we want to prove is that there exists a unique 2-morphism

$$\theta : T \Rightarrow T' : \mathbb{X} \rightarrow \mathbb{C} \times \mathbb{D}$$

such that

$$\theta \bullet^0 \Pi_{\mathbb{C}} = \alpha, \quad \theta \bullet^0 \Pi_{\mathbb{D}} = \beta, \quad (3.11)$$

First let us say that T and T' are determined by 1-dimensional universal property:

$$T \text{ is such that (and univocally determined by) } \begin{cases} T \bullet^0 \Pi_{\mathbb{C}} = A \\ T \bullet^0 \Pi_{\mathbb{D}} = B \end{cases},$$

$$T' \text{ is such that (and univocally determined by) } \begin{cases} T' \bullet^0 \Pi_{\mathbb{C}} = A' \\ T' \bullet^0 \Pi_{\mathbb{D}} = B' \end{cases}.$$

More explicitly, for every pair of objects x_0, x'_0 in \mathbb{X} ,

$$T_0(x_0) = (A_0(x_0), B_0(x_0))$$

and

$$\begin{array}{ccc} \mathbb{X}_1(x_0, x'_0) & \xrightarrow{T_1^{x_0, x'_0}} & [\mathbb{C} \times \mathbb{D}]_1^{x_0, x'_0}((Ax_0, Bx_0), (Ax'_0, Bx'_0)) \\ & \searrow \langle A_1^{x_0, x'_0}, B_1^{x_0, x'_0} \rangle & \parallel \text{def} \\ & & \mathbb{C}_1(Ax_0, Ax'_0) \times \mathbb{D}(Bx_0, Bx'_0) \end{array}$$

Similarly for T' : $T'_0(x_0) = (A'_0(x_0), B'_0(x_0))$ and $T'_1^{x_0, x'_0} = \langle A'_1{}^{x_0, x'_0}, B'_1{}^{x_0, x'_0} \rangle$. Then, $\theta = \langle \theta_0, \theta_1 \rangle$ is given by:

$$\theta_0(x_0) = (\alpha_0(x_0), \beta_0(x_0))$$

and $\theta_1^{x_0, x'_0}$ is given by the universal property of products in $(n-1)\mathbf{Cat}$.

In fact, a suitable θ_1 would fit in the following diagram:

$$\begin{array}{ccccc} & & \mathbb{X}_1(x_0, x'_0) & & \\ & \swarrow T_1^{x_0, x'_0} & & \searrow T'_1{}^{x_0, x'_0} & \\ [\mathbb{C} \times \mathbb{D}]_1((Ax_0, Bx_0), (Ax'_0, Bx'_0)) & \xleftarrow{\quad ? \quad} & & & [\mathbb{C} \times \mathbb{D}]_1((A'x_0, B'x_0), (A'x'_0, B'x'_0)) \\ & \searrow - \circ \theta_{x'_0} & & \swarrow \theta_{x_0} \circ - & \\ & & [\mathbb{C} \times \mathbb{D}]_1((Ax_0, Bx_0), (A'x'_0, B'x'_0)) & & \end{array}$$

Spelling out the definitions, this may be written:

$$\begin{array}{ccc}
 & \mathbb{X}_1(x_0, x'_0) & \\
 \swarrow \langle A_1^{x_0, x'_0}, B_1^{x_0, x'_0} \rangle & & \searrow \langle A'_1{}^{x_0, x'_0}, B'_1{}^{x_0, x'_0} \rangle \\
 \mathbb{C}_1(Ax_0, Ax'_0) \times \mathbb{D}_1(Bx_0, Bx'_0) & \xleftarrow{\theta_1^{x_0, x'_0}} & \mathbb{C}_1(A'x_0, A'x'_0) \times \mathbb{D}_1(B'x_0, B'x'_0) \\
 \searrow (-\circ \alpha_{x'_0}) \times (-\circ \beta_{x'_0}) & & \swarrow (\alpha_{x_0} \circ -) \times (\beta_{x_0} \circ -) \\
 & \mathbb{C}_1(Ax_0, A'x'_0) \times \mathbb{D}_1(Bx_0, B'x'_0) &
 \end{array}$$

hence we are allowed to define

$$\theta_1^{x_0, x'_0} = \langle \alpha_1^{x_0, x'_0}, \beta_1^{x_0, x'_0} \rangle,$$

and this choice would satisfy the universal property of products..

Concerning the first of the (3.11), for every pair of objects x_0, x'_0 in \mathbb{X}

$$[\theta \bullet^0 \Pi_{\mathbb{C}}]_0(x_0) = [\Pi_{\mathbb{C}}]_0(\theta_{x_0}) = \pi_{\mathbb{C}_0 \times \mathbb{D}_0}^{\mathbb{C}_0 \times \mathbb{D}_0}(\alpha_{x_0}, \beta_{x_0}) = \alpha_{x_0}$$

and also

$$\begin{aligned}
 [\theta \bullet^0 \Pi_{\mathbb{C}}]_1^{x_0, x'_0} &= \theta_1^{x_0, x'_0} \bullet^0 [\Pi_{\mathbb{C}}]_1^{(Ax_0, Bx_0), (A'x'_0, B'x'_0)} = \\
 &= \langle \alpha_1^{x_0, x'_0}, \beta_1^{x_0, x'_0} \rangle \bullet^0 \Pi_{\mathbb{C}_1(Ax_0, A'_0 x'_0)} = \alpha_1^{x_0, x'_0}
 \end{aligned}$$

where everything comes directly from definitions, but the last equality which is given by the universal property defining $\theta_1^{x_0, x'_0}$ in $(n-1)\mathbf{Cat}$. The second of the (3.11) can be proved the same way.

Moreover, such a θ is unique. For, if another

$$\eta : T \Rightarrow T' : \mathbb{X} \rightarrow \mathbb{C} \times \mathbb{D}$$

is such that

$$\eta \bullet^0 \Pi_{\mathbb{C}} = \alpha, \quad \eta \bullet^0 \Pi_{\mathbb{D}} = \beta,$$

then equations above determine η_0 on objects (since a map to a product, $\mathbb{C}_0 \times \mathbb{D}_0$, is determined by its projections), hence it is equal to θ_0 . On the other side, for objects x_0, x'_0 in \mathbb{X} , and $\eta_1^{x_0, x'_0}$ is given considering its composition with projections by universality of the product in $(n-1)\mathbf{Cat}$, hence it is equal to $\theta_1^{x_0, x'_0}$.

To conclude this section, we will see that the just defined pair $[\theta_0, \theta_1^{-, -}]$ is indeed a 2-morphism, that is, it obeys units and composition axioms for n -transformations.

To this purpose, let us chose a triple of objects x_0, x'_0 and x''_0 of \mathbb{X} , and let us consider the following diagram:

$$\begin{array}{ccc}
& [x_0, x'_0] \times [x'_0, x''_0] & \\
\swarrow^{id \times L^{x'_0, x''_0}} & & \searrow^{L^{x'_0, x''_0} \times id} \\
[x_0, x'_0] \times [Ax'_0, A'x''_0] \times [Bx'_0, B'x''_0] & \xleftarrow{id \times \theta_1^{x'_0, x''_0}} & [Ax_0, A'x'_0] \times [Bx_0, B'x'_0] \times [x'_0, x''_0] \\
\downarrow \langle A_1^{x_0, x'_0}, B_1^{x_0, x'_0} \rangle \times id & & \downarrow id \times \langle A_1^{x'_0, x''_0}, B_1^{x'_0, x''_0} \rangle \\
[Ax_0, Ax'_0] \times [Bx_0, Bx'_0] \times [Ax'_0, A'x''_0] \times [Bx'_0, B'x''_0] & & [Ax_0, A'x'_0] \times [Bx_0, B'x'_0] \times [A'x'_0, A'x''_0] \times [B'x'_0, B'x''_0] \\
\downarrow \tau & & \downarrow \tau \\
[Ax_0, Ax'_0] \times [Ax'_0, A'x''_0] \times [Bx_0, Bx'_0] \times [Bx'_0, B'x''_0] & & [Ax_0, A'x'_0] \times [A'x'_0, A'x''_0] \times [Bx_0, B'x'_0] \times [B'x'_0, B'x''_0] \\
\searrow \mathbb{C}_\circ \times \mathbb{D}_\circ & & \swarrow \mathbb{C}_\circ \times \mathbb{D}_\circ \\
& [Ax_0, A'x''_0] \times [Bx_0, B'x''_0] &
\end{array}$$

where

$$\begin{aligned}
L^{x,y} &= \langle A_1^{x,y}, B_1^{x,y} \rangle \left((- \circ \alpha_y) \times (- \circ \beta_y) \right) \\
L^{x,y} &= \langle A_1^{x,y}, B_1^{x,y} \rangle \left((\alpha_x \circ -) \times (\beta_x \circ -) \right)
\end{aligned}$$

Now, considering only the right branch of the diagram, where Δ 's express diagonal morphisms, the following equations hold by product interchange properties:

$$\begin{aligned}
& (\theta_1^{x_0, x'_0} \times id) \bullet^0 (id \times \langle A_1^{x'_0, x''_0}, B_1^{x'_0, x''_0} \rangle) \tau (\mathbb{C}_\circ \times \mathbb{D}_\circ) = \\
&= (\langle \alpha_1^{x_0, x'_0}, \beta_1^{x_0, x'_0} \rangle \times id) \bullet^0 (id \times \langle A_1^{x'_0, x''_0}, B_1^{x'_0, x''_0} \rangle) \tau (\mathbb{C}_\circ \times \mathbb{D}_\circ) \\
&= (\Delta \times id) \bullet^0 (\alpha_1^{x_0, x'_0} \times \beta_1^{x_0, x'_0} \times id) \bullet^0 (id \times \Delta) (id \times A_1^{x'_0, x''_0} \times B_1^{x'_0, x''_0}) \tau (\mathbb{C}_\circ \times \mathbb{D}_\circ) \\
&= (\Delta \times \Delta) \tau \bullet^0 (\alpha_1^{x_0, x'_0} \times id \times \beta_1^{x_0, x'_0} \times id) \bullet^0 (id \times A_1^{x'_0, x''_0} \times id \times B_1^{x'_0, x''_0}) (\mathbb{C}_\circ \times \mathbb{D}_\circ) \\
&= (\Delta \times \Delta) \tau \bullet^0 \left((\alpha_1^{x_0, x'_0} \times A_1^{x'_0, x''_0} \bullet^0 (\mathbb{C}_\circ)) \times (\beta_1^{x_0, x'_0} \times B_1^{x'_0, x''_0} \bullet^0 (\mathbb{D}_\circ)) \right)
\end{aligned}$$

Similarly, the left branch gives:

$$\begin{aligned}
& (id \times \theta_1^{x'_0, x''_0}) \bullet^0 (\langle A_1^{x'_0, x''_0}, B_1^{x'_0, x''_0} \rangle \times id) \tau (\mathbb{C}_\circ \times \mathbb{D}_\circ) = \\
&= (\Delta \times \Delta) \tau \bullet^0 \left((A_1^{x_0, x'_0} \times \alpha_1^{x'_0, x''_0} \bullet^0 (\mathbb{C}_\circ)) \times (B_1^{x_0, x'_0} \times \beta_1^{x'_0, x''_0} \bullet^0 (\mathbb{D}_\circ)) \right)
\end{aligned}$$

Hence the diagram above, being the vertical composite of the two branches, may be rewritten applying rule (L4):

$$\begin{aligned}
&= (\Delta \times \Delta) \tau \bullet^0 \left[\left((\alpha_1^{x_0, x'_0} \times A'_1 x'_0, x''_0 \bullet^0 (\mathbb{C}_o)) \bullet^1 (A_1 x_0, x'_0 \times \alpha_1^{x'_0, x''_0} \bullet^0 (\mathbb{C}_o)) \right) \times \right. \\
&\quad \left. \times \left((\beta_1^{x_0, x'_0} \times B'_1 x'_0, x''_0 \bullet^0 (\mathbb{D}_o)) \bullet^1 (B_1 x_0, x'_0 \times \beta_1^{x'_0, x''_0} \bullet^0 (\mathbb{D}_o)) \right) \right]
\end{aligned}$$

by composition axiom of α and β , and product properties, we get the result:

$$\begin{aligned}
&= (\Delta \times \Delta) \tau \bullet^0 \left(((\mathbb{X}_o) \bullet^0 \alpha_1^{x_0, x''_0}) \times ((\mathbb{X}_o) \bullet^0 \beta_1^{x_0, x''_0}) \right) \\
&= (\mathbb{X}_o) \Delta \bullet^0 (\alpha_1^{x_0, x''_0} \times \beta_1^{x_0, x''_0}) \\
&= (\mathbb{X}_o) \bullet^0 \langle \alpha_1^{x_0, x''_0}, \beta_1^{x_0, x''_0} \rangle \\
&= (\mathbb{X}_o) \bullet^0 \theta_1^{x_0, x''_0}
\end{aligned}$$

Furthermore, for an object x_0 , consider the following diagram:

$$\begin{array}{ccc}
& \mathbb{I} & \\
& \downarrow u(x_0) & \\
& [x_0, x_0] & \\
L^{x_0, x_0} \swarrow & \theta_1^{x_0, x_0} & \searrow L'^{x_0, x_0} \\
& [Ax_0, A'x_0] \times [Bx_0, B'x_0] &
\end{array}$$

then, as for the composition axiom,

$$\begin{aligned}
u(x_0) \bullet^0 \theta_1^{x_0, x_0} &= u(x_0) \circ \langle \alpha_1^{x_0, x_0}, \beta_1^{x_0, x_0} \rangle \\
&= u(x_0) \Delta \bullet^0 (\alpha_1^{x_0, x_0} \times \beta_1^{x_0, x_0}) \\
&= \Delta u(x_0) \bullet^0 (\alpha_1^{x_0, x_0} \times \beta_1^{x_0, x_0}) \\
&= \Delta \bullet^0 ((u(x_0) \bullet^0 \alpha_1^{x_0, x_0}) \times (u(x_0) \bullet^0 \beta_1^{x_0, x_0})) \\
&\stackrel{(*)}{=} \Delta \bullet^0 (Id_{[\alpha_{x_0}]} \times Id_{[\beta_{x_0}]}) \\
&= Id_{\langle [\alpha_{x_0}], [\beta_{x_0}] \rangle} = Id_{[\theta_{x_0}]}
\end{aligned}$$

where $(*)$ is given by unit axiom of α and β .

3.6 The standard h -pullback in $n\mathbf{Cat}$

In this section we give a construction that will be of fundamental importance for the development of the theory. The idea is to generalize a classical homotopical construction [Mat76b] to n -categories, or better to n -groupoids, where homotopical aspects are more than a mere suggestion.

We start considering the following h -pullbacks reference diagram.

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{Q} & \mathbb{C} \\ P \downarrow & \nearrow \varepsilon & \downarrow G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array}$$

For $n=0$, classical pullback in \mathbf{Set} is an instance of h -pullback, with 2-morphism ε being an identity.

In fact the category of sets and maps is (seen as) the 2-trivial sesqui-category $0\mathbf{Cat}$, and indeed, only condition (1) and (2) survive. Hence, in the next sections, we will suppose integer $n > 0$ been given.

We exhibit a recursive construction of the standard h -pullback satisfying *Universal Property 2.12*.

We will give \mathbb{P} in the form $(\mathbb{P}_0, \mathbb{P}_1^-, \cdot)$.

The set \mathbb{P}_0 is the following limit in \mathbf{Set} (that yields indeed, also the object-components of F , G and ε):

$$\begin{array}{ccccc} & & \mathbb{P}_0 & & \\ & \swarrow P_0 & \vdots \varepsilon_0 & \searrow Q_0 & \\ \mathbb{A}_0 & & \mathbb{B}_1 & & \mathbb{C}_0 \\ & \searrow F_0 & \swarrow s \quad \searrow t & \swarrow G_0 & \\ & & \mathbb{B}_0 & & \mathbb{B}_0 \end{array}$$

Here \mathbb{B}_1 is the disjoint union

$$\coprod_{b_0, b'_0 \in \mathbb{B}_0} [\mathbb{B}_1(b_0, b'_0)]_0,$$

and s, t are *source* and *target* maps of 1-cells. More explicitly,

$$\mathbb{P}_0 = \{(a_0, b_1, c_0) \mid s.t. \ a_0 \in \mathbb{A}_0, c_0 \in \mathbb{C}_0, b_1 : Fa_0 \rightarrow Gc_0 \in \mathbb{B}_1\}$$

$$P_0((a_0, b_1, c_0)) = a_0, \quad Q_0((a_0, b_1, c_0)) = c_0, \quad \varepsilon_0((a_0, b_1, c_0)) = b_1$$

Let us fix two element of \mathbb{P}_0 :

$$p_0 = (a_0, b_1, c_0), \quad p'_0 = (a'_0, b'_1, c'_0).$$

The hom- $(n-1)$ category $\mathbb{P}_1(p_0, p'_0)$ is granted by the following h-pullback in $(n-1)\mathbf{Cat}$:

$$\begin{array}{ccccc}
 \mathbb{P}_1(p_0, p'_0) & \xrightarrow{Q_1^{p_0, p'_0}} & \mathbb{C}_1(c_0, c'_0) & & \\
 \downarrow P_1^{p_0, p'_0} & \swarrow \varepsilon_1^{p_0, p'_0} & \downarrow G_1^{c_0, c'_0} & & \\
 \mathbb{A}_1(a_0, a'_0) & \xrightarrow{F_1^{a_0, a'_0}} & \mathbb{B}_1(Fa_0, Fa'_0) & \xrightarrow{- \circ b'_1} & \mathbb{B}_1(Fa_0, Gc'_0) \\
 & & & & \downarrow b_1 \circ - \\
 & & & & \mathbb{B}_1(Gc_0, Gc'_0)
 \end{array}$$

Proposition 3.5. *The pair $(\mathbb{P}_0, \mathbb{P}_1^-, -)$ yields a n -category.*

In order to prove the statement, we need to show constructions for composition and units, and to prove that they satisfies n -category axioms.

3.6.1 Composition

Suppose we are given three elements of \mathbb{P}_0

$$p_0 = (a_0, b_1, c_0), \quad p'_0 = (a'_0, b'_1, c'_0), \quad p''_0 = (a''_0, b''_1, c''_0).$$

One defines

$$\circ^0 : \mathbb{P}_1(p_0, p'_0) \times \mathbb{P}_1(p'_0, p''_0) \rightarrow \mathbb{P}_1(p_0, p''_0)$$

by means of the universal property of pullback in $(n-1)\mathbf{Cat}$.

In fact, as $\mathbb{P}_1(p_0, p''_0)$ is a pullback, we can consider the four-tuple

$$\left(\mathbb{P}_1(p_0, p'_0) \times \mathbb{P}_1(p'_0, p''_0), \quad P_1^{p_0, p'_0} \times P_1^{p'_0, p''_0}, \quad Q_1^{p_0, p'_0} \times Q_1^{p'_0, p''_0}, \quad \varepsilon^{p_0, p'_0, p''_0} \right)$$

where the $(n-1)$ -natural transformation $\varepsilon^{p_0, p'_0, p''_0}$ is the composite shown below:

$$\begin{array}{c}
\begin{array}{ccccc}
& [p_0, p'_0] \times [a'_0, a''_0] & \xleftarrow{id \times P_1} & [p_0, p'_0] \times [p'_0, p''_0] & \xrightarrow{Q_1 \times id} & [c_0, c'_0] \times [p'_0, p''_0] \\
& \swarrow P_1 \times id & & \swarrow id \times Q_1 & & \searrow id \times Q_1 \\
[a_0, a'_0] \times [a'_0, a''_0] & & [p_0, p'_0] \times [c'_0, c''_0] & & [a_0, a'_0] \times [p'_0, p''_0] & & [c_0, c'_0] \times [c'_0, c''_0] \\
& \downarrow id \times F_1 & \swarrow id \times \varepsilon_1^{p'_0, p''_0} & \downarrow id \times G_1 & \downarrow F_1 \times id & \swarrow \varepsilon_1^{p_0, p'_0} \times id & \downarrow G_1 \times id \\
[p_0, p'_0] \times [Fa'_0, Fa''_0] & & [p_0, p'_0] \times [Gc'_0, Gc''_0] & & [Fa_0, Fa'_0] \times [p'_0, p''_0] & & [Gc_0, Gc'_0] \times [p'_0, p''_0] \\
& \downarrow id \times (-\circ b'_1) & \swarrow id \times (b'_1 \circ -) & & \downarrow (-\circ b'_1) \times id & & \downarrow (b_1 \circ -) \times id \\
[p_0, p'_0] \times [Fa'_0, Gc'_0] & & & & [Fa_0, Gc'_0] \times [p'_0, p''_0] & & \\
& \downarrow P_1 \times id & & & \downarrow id \times Q_1 & & \\
[a_0, a'_0] \times [Fa'_0, Gc'_0] & & & & [Fa_0, Gc'_0] \times [c'_0, c''_0] & & \\
& \swarrow F_1 \times id & & \swarrow id \times G_1 & & & \\
[Fa_0, Fa'_0] \times [Fa'_0, Gc'_0] & & [Fa_0, Gc'_0] \times [Gc'_0, Gc''_0] & & & & \\
& \searrow \circ & \swarrow \circ & & & & \\
[Fa_0, Fa'_0] & \xrightarrow{-\circ b'_1} & [Fa_0, Gc'_0] & \xleftarrow{b_1 \circ -} & [Gc_0, Gc'_0] & \xrightarrow{G_1} & [c_0, c'_0]
\end{array}
\end{array}$$

(3.12)

Dotted outer border is clearly equal to continuous inner border, hence, by universal property in $(n-1)\mathbf{Cat}$, there exists a unique

$$\mathbb{P}_o^0 : \mathbb{P}_1(p_0, p'_0) \times \mathbb{P}_1(p'_0, p''_0) \longrightarrow \mathbb{P}_1(p_0, p''_0) \quad (3.13)$$

such that two squares below commute

$$\begin{array}{ccccc}
\mathbb{A}_1(a_0, a'_0) \times \mathbb{A}_1(a'_0, a''_0) & \xleftarrow{P_1 \times P_1} & \mathbb{P}_1(p_0, p'_0) \times \mathbb{P}_1(p'_0, p''_0) & \xrightarrow{Q_1 \times Q_1} & \mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c''_0) \\
\downarrow \mathbb{A}_o & & \downarrow \mathbb{P}_o & & \downarrow \mathbb{C}_o \\
\mathbb{A}_1(a_0, a''_0) & \xleftarrow{P_1} & \mathbb{P}_1(p_0, p''_0) & \xrightarrow{Q_1} & \mathbb{C}_1(c_0, c''_0)
\end{array} \quad (3.14)$$

and

$$\begin{array}{ccc}
 \mathbb{P}_1(p_0, p'_0) \times \mathbb{P}_1(p'_0, p''_0) & & \mathbb{P}_1(p_0, p'_0) \times \mathbb{P}_1(p'_0, p''_0) \\
 \downarrow \mathbb{P}_\circ & & \downarrow \varepsilon_1^{p_0, p'_0, p''_0} \\
 \mathbb{P}_1(p_0, p''_0) & = & \mathbb{P}_1(p_0, p''_0) \\
 \downarrow \varepsilon_1^{p_0, p''_0} & & \downarrow \varepsilon_1^{p_0, p'_0, p''_0} \\
 \mathbb{B}_1(Fa_0, Gc''_0) & & \mathbb{B}_1(Fa_0, Gc''_0)
 \end{array} \quad (3.15)$$

Lemma 3.6. *Composition \mathbb{P}_\circ^0 defined above is associative, i.e. the diagram below commutes in $(n-1)\mathbf{Cat}$, for every four-tuple $(p_0, p'_0, p''_0, p'''_0)$ of elements of \mathbb{P}_0 :*

$$\begin{array}{ccc}
 \mathbb{P}_1(p_0, p'_0) \times \mathbb{P}_1(p'_0, p''_0) \times \mathbb{P}_1(p''_0, p'''_0) & \xrightarrow{id \times \mathbb{P}_\circ} & \mathbb{P}_1(p_0, p'_0) \times \mathbb{P}_1(p'_0, p'''_0) \\
 \downarrow \mathbb{P}_\circ \times id & & \downarrow \mathbb{P}_\circ \\
 \mathbb{P}_1(p_0, p''_0) \times \mathbb{P}_1(p''_0, p'''_0) & \xrightarrow{\mathbb{P}_\circ} & \mathbb{P}_1(p_0, p'''_0)
 \end{array} \quad (3.16)$$

Proof. Let us consider the diagrams

$$\begin{array}{ccccc}
 [p_0, p'_0] \times [p'_0, p''_0] \times [p''_0, p'''_0] & \xrightarrow{Q_1 \times Q_1 \times Q_1} & [c_0, c'_0] \times [c'_0, c''_0] \times [c''_0, c'''_0] \\
 \downarrow P_1 \times P_1 \times P_1 & \searrow K & \downarrow \Xi(\mathbb{C}) \\
 [a_0, a'_0] \times [a'_0, a''_0] \times [a''_0, a'''_0] & \xrightarrow{\Xi(\mathbb{A})} & [a_0, a'''_0] & \xrightarrow{\quad} & [Fa_0, Gc'''_0] \\
 & & \downarrow P_1 & \swarrow \varepsilon_1^{p_0, p'_0, p''_0} & \downarrow \\
 & & [p_0, p'''_0] & \xrightarrow{Q_1} & [c_0, c'''_0]
 \end{array}$$

where morphism $\Xi(\mathcal{X})$ can be either the composite $(\mathcal{X}_\circ \times id)\mathcal{X}_\circ$ or $(id \times \mathcal{X}_\circ)\mathcal{X}_\circ$, with \mathcal{X} being \mathbb{A} or \mathbb{C} .

The idea of the proof is to use again universal property in $(n-1)\mathbf{Cat}$ to get unique

$$K : \mathbb{P}_1(p_0, p'_0) \times \mathbb{P}_1(p'_0, p''_0) \times \mathbb{P}_1(p''_0, p'''_0) \rightarrow \mathbb{P}_1(p_0, p'''_0)$$

that coincides with both composites of diagram 3.16. To this end, it suffices to show that the four-tuples

$$\left(\begin{array}{c}
 \mathbb{P}_1(p_0, p'_0) \times \mathbb{P}_1(p'_0, p''_0) \times \mathbb{P}_1(p''_0, p'''_0) \\
 (P_1 \times P_1 \times P_1)(\mathbb{A}_\circ \times id)^{\mathbb{A}_\circ} \\
 (Q_1 \times Q_1 \times Q_1)(\mathbb{C}_\circ \times id)^{\mathbb{C}_\circ} \\
 (\mathbb{P}_\circ \times id)^{\mathbb{P}_\circ} \varepsilon_1^{p_0, p'_0, p''_0}
 \end{array} \right)$$

and

$$\begin{pmatrix} \mathbb{P}_1(p_0, p'_0) \times \mathbb{P}_1(p'_0, p''_0) \times \mathbb{P}_1(p''_0, p'''_0) \\ (P_1 \times P_1 \times P_1)(id \times \mathbb{A}_\circ) \mathbb{A}_\circ \\ (Q_1 \times Q_1 \times Q_1)(id \times \mathbb{C}_\circ) \mathbb{C}_\circ \\ (id \times \mathbb{P}_\circ) \mathbb{P}_\circ \circ \varepsilon_1^{p_0, p'''_0} \end{pmatrix}$$

are equal.

First components are identical.

Equality of second components amounts to the associativity axiom for 0-composition in \mathbb{A} .

Equality of third components amounts to the associativity axiom for 0-composition in \mathbb{C} .

What remains to prove is equality of fourth components. The starting point is the 2-morphism $(\mathbb{P}_\circ \times id) \bullet^0 \mathbb{P}_\circ \bullet^0 \varepsilon_1^{p_0, p'''_0}$, where

$$\varepsilon_1^{p_0, p'''_0} : b_1 \circ [QG]_1(-) \Rightarrow [PF]_1(-) \circ b_1'''$$

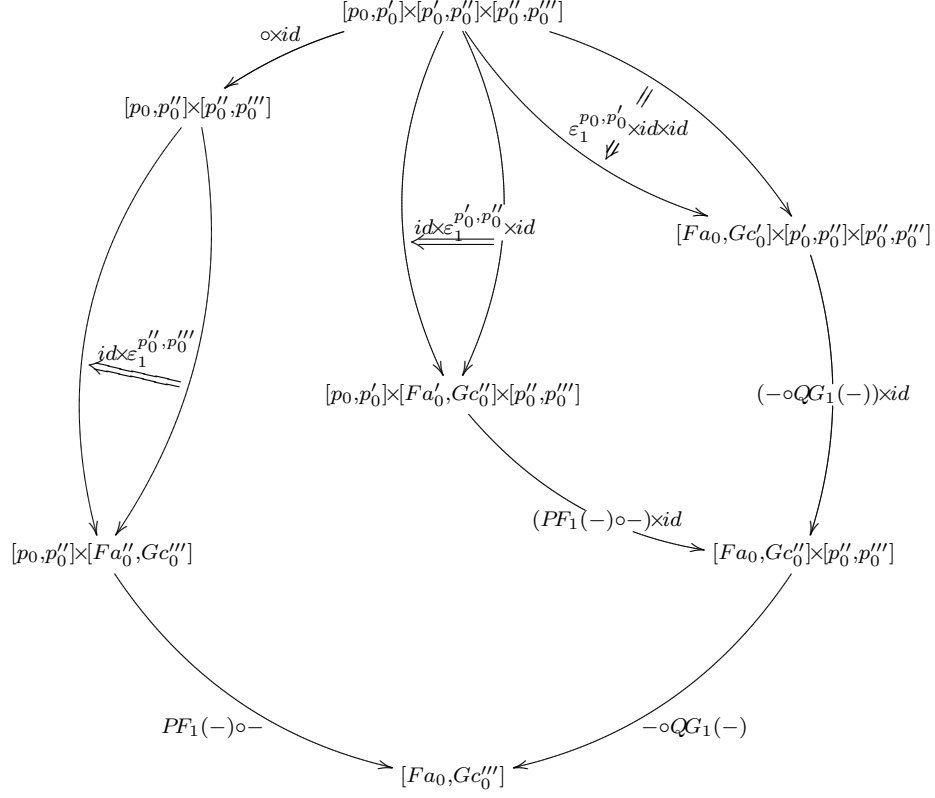
This may be visualized as a diagram:

$$\begin{array}{ccccc} & & [p_0, p'_0] \times [p'_0, p''_0] \times [p''_0, p'''_0] & & \\ & & \downarrow \circ \times id & & \\ & & [p_0, p'_0] \times [p'_0, p'''_0] & & \\ & & \downarrow \circ & & \\ & & [p_0, p'''_0] & & \\ & \swarrow P_1 & & \searrow Q_1 & \\ [a_0, a'''_0] & & & & [c_0, c'''_0] \\ \downarrow F_1 & & \xleftarrow{\varepsilon_1^{p_0, p'''_0}} & & \downarrow G_1 \\ [Fa_0, Fa'''_0] & & & & [Gc_0, Gc'''_0] \\ & \searrow - \circ b_1''' & & \swarrow b_1 \circ - & \\ & & [Fc_0, Gc'''_0] & & \end{array}$$

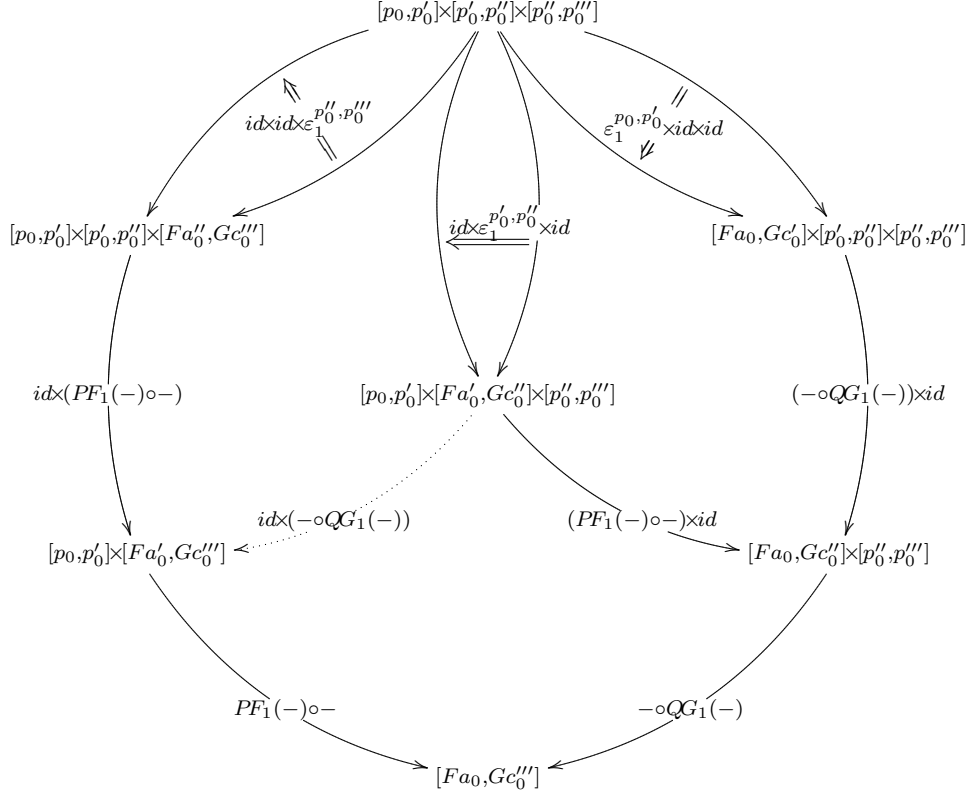
that, applying (3.12) and (3.15) to $\varepsilon_1^{p_0, p'''_0}$, equals to

$$\begin{array}{c}
[p_0, p'_0] \times [p'_0, p''_0] \times [p''_0, p'''_0] \\
\downarrow \circ \times id \\
\begin{array}{ccccc}
[p_0, p''_0] \times [a''_0, a'''_0] & \xleftarrow{id \times P_1} & [p_0, p'_0] \times [p''_0, p'''_0] & \xrightarrow{Q_1 \times id} & [c_0, c''_0] \times [p'_0, p'''_0] \\
\downarrow id \times F_1 & & \swarrow id \times Q_1 \quad \searrow P_1 \times id & & \downarrow G_1 \times id \\
[p_0, p'_0] \times [Fa''_0, Fa'''_0] & \xleftarrow{id \times \varepsilon_1^{p'_0, p''_0}} & [p_0, p'_0] \times [c''_0, c'''_0] & & [a_0, a''_0] \times [p''_0, p'''_0] \\
\downarrow id \times (- \circ b_1''') & & \downarrow id \times G_1 & & \downarrow F_1 \times id \\
[p_0, p'_0] \times [Fa''_0, Gc''_0] & \xleftarrow{id \times (b_1'' \circ -)} & [p_0, p'_0] \times [Gc''_0, Gc'''_0] & & [Fa_0, Fa''_0] \times [p''_0, p'''_0] \\
\downarrow P_1 \times id & & & & \downarrow (- \circ b_1'') \times id \\
[a_0, a''_0] \times [Fa''_0, Gc''_0] & & & & [Fa_0, Gc''_0] \times [p''_0, p'''_0] \\
\downarrow F_1 \times id & & & & \downarrow id \times Q_1 \\
[Fa_0, Fa''_0] \times [Fa''_0, Gc''_0] & & & & [Fa_0, Gc''_0] \times [c''_0, c'''_0] \\
\downarrow \circ & & & & \downarrow id \times G_1 \\
[Fa_0, Gc''_0] & & & & [Fa_0, Gc''_0] \times [Gc''_0, Gc'''_0] \\
\downarrow \circ & & & & \\
[Fa_0, Gc'''_0] & & & &
\end{array}
\end{array}$$

Now we can apply (3.12) again, and (3.15) to the right-hand side of the diagram, to express $\varepsilon_1^{p_0, p'_0}$ in terms of $\varepsilon_1^{p_0, p'_0}$ and $\varepsilon_1^{p'_0, p''_0}$.



Moreover, in order to shift the 2-morphism $\varepsilon_1^{p''_0, p'''_0}$ up, we apply *product interchange rules* to the left-hand side. What we get is the diagram:



The dotted arrow fits the diagram properly, making the two regions commute. Hence the whole diagram is perfectly symmetric, and calculations may be carried on doing the steps in reverse order, and gain the result. \square

3.6.2 Units

As for composition, we get unit morphisms by universal property in $(n-1)\mathbf{Cat}$.

Suppose an element p_0 of \mathbb{P}_0 is fixed.

$$\begin{array}{ccc}
 \mathbb{I} & \xrightarrow{u(c_0)} & \mathbb{C}_1(c_0, c_0) \\
 u(a_0) \downarrow & & \downarrow b_1 \circ G(-) \\
 \mathbb{A}_1(a_0, a_0) & \xrightarrow{F(-) \circ b_1} & \mathbb{B}_1(Fa_0, Gc_0)
 \end{array}$$

The square above commutes, hence it is an identity 2-morphism, over the same base defining $\varepsilon_1^{p_0, p_0}$, and this implies the existence of a unique

$$\mathbb{P}u(p_0) : \mathbb{I} \longrightarrow \mathbb{P}_1(p_0, p_0) .$$

Lemma 3.7. *Units defined above are neutral w.r.t. 0-composition, i.e., for every pair p_0, p'_0 , in \mathbb{P}_0 , (L) and (R) commute.*

$$\begin{array}{ccc}
 \mathbb{P}_1(p_0, p'_0 \times \mathbb{I}) & \xrightarrow{id \times u(p'_0)} & \mathbb{P}_1(p_0, p'_0) \times \mathbb{P}_1(p'_0, p'_0) \\
 \rho \uparrow \cong & (R) & \downarrow \circ \\
 \mathbb{P}_1(p_0, p'_0) & \xrightarrow{id} & \mathbb{P}_1(p_0, p'_0) \\
 \lambda \downarrow \cong & (L) & \uparrow \circ \\
 \mathbb{I} \times \mathbb{P}_1(p_0, p'_0) & \xrightarrow{u(p_0) \times id} & \mathbb{P}_1(p_0, p_0) \times \mathbb{P}_1(p_0, p'_0)
 \end{array}$$

Proof. We show only the commutativity of (L), the other being similar. Hence let us consider the composition

$$\begin{array}{ccc}
 & [p_0, p'_0] & \\
 & \cong \downarrow \lambda & \\
 & \mathbb{I} \times [p_0, p'_0] & \\
 & u(p_0) \times id \downarrow & \\
 P_1 \swarrow & [p_0, p_0] \times [p_0, p'_0] & \searrow Q_1 \\
 & \downarrow \circ & \\
 & [p_0, p'_0] & \\
 P_1 \swarrow & & \searrow Q_1 \\
 [a_0, a'_0] & \xleftarrow{\varepsilon_1^{p_0, p'_0}} & [c_0, c'_0] \\
 F(-) \circ b'_1 \searrow & & \swarrow b_1 \circ G(-) \\
 & [Fa_0, Gc'_0] &
 \end{array} \tag{3.17}$$

It is easy to see that both sides commute with dotted arrows. In fact

$$\begin{aligned}
 \lambda(\mathbb{P}u(p_0) \times id)(\mathbb{P}\circ)Q_1 & \stackrel{(i)}{=} \lambda(\mathbb{P}u(p_0) \times id)(Q_1 \times Q_1)(\mathbb{C}\circ) \\
 & \stackrel{(ii)}{=} \lambda(\mathbb{C}u(c_0) \times Q_1)(\mathbb{C}\circ) \\
 & \stackrel{(iii)}{=} Q_1
 \end{aligned}$$

where (i) holds by (3.14), (ii) by definition of $\mathbb{P}u$, (iii) by the *unit axiom* in \mathbb{C} . Similarly for the left-hand side.

If one shows that composition above is equal to $\varepsilon_1^{p_0, p'_0}$, the universal property of pullbacks implies (L). Now we can reformulate it with the help of (3.12)

and (3.15):

$$\begin{array}{c}
 [p_0, p'_0] \\
 \downarrow \cong \lambda \\
 \mathbb{I} \times [p_0, p'_0] \\
 \downarrow u(p_0) \times id \\
 [p_0, p_0] \times [p_0, p'_0] \\
 \swarrow id \times \varepsilon_1^{p_0, p'_0} \quad \searrow [b_1] \times id \\
 [p_0, p_0] \times [Fa_0, Gc'_0] \quad [Fa_0, Gc_0] \times [p_0, p'_0] \\
 \searrow PF_1(-) \circ - \quad \swarrow - \circ QG_1(-) \\
 [Fa_0, Gc'_0]
 \end{array}$$

where we have somehow abusively replaced the identity 2-morphism on the right hand side, with its source (= target) 1-morphism. Hence all the right-hand side, being an identity, may be cancelled. Finally, by *product interchange*

$$\begin{array}{ccc}
 \begin{array}{c}
 [p_0, p'_0] \\
 \downarrow \cong \lambda \\
 \mathbb{I} \times [p_0, p'_0] \\
 \downarrow id \times \varepsilon_1^{p_0, p'_0} \\
 \mathbb{I} \times [Fa_0, Gc'_0] \\
 \downarrow u(Fa_0) \circ - \\
 [Fa_0, Gc'_0]
 \end{array} & \stackrel{(1)}{=} & \begin{array}{c}
 [p_0, p'_0] \\
 \downarrow \varepsilon_1^{p_0, p'_0} \\
 [Fa_0, Gc'_0] \\
 \downarrow \cong \lambda \\
 \mathbb{I} \times [Fa_0, Gc'_0] \\
 \downarrow u(Fa_0) \circ - \\
 [Fa_0, Gc'_0]
 \end{array} \\
 & & \stackrel{(2)}{=} \varepsilon_1^{p_0, p'_0}
 \end{array}$$

where (1) holds by naturality of $\lambda_{(-)} : (-) \Rightarrow \mathbb{I} \times (-)$, and (2) by neutral identities in \mathbb{B} . \square

3.6.3 Projections and ε

So far we proved that the pair $\mathbb{P} = (\mathbb{P}_0, \mathbb{P}_1^-, \cdot)$ is indeed a n -category. In order to show it is a part of a pullback four-tuple, we should prove that

$$P = (P_0, P_1^-, \cdot), \quad Q = (Q_0, Q_1^-, \cdot), \quad \varepsilon = (\varepsilon_0, \varepsilon_1^-, \cdot)$$

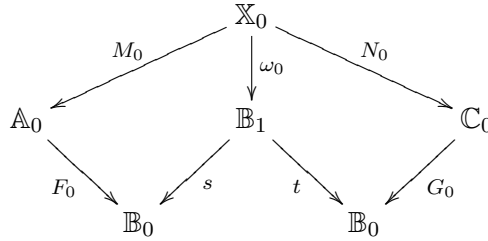
produced in stating the definitions above, constitute respectively two n -category morphisms and one 2-morphism. But this has been already proved throughout the last sections. In fact, universal definition of 0-composition above reveals that this is just the one that makes P , Q and ε functorial. Similarly, universal definition of 0-units reveals that these are just the ones that make P , Q and ε functorial.

3.6.4 Universal property

The final step in proving that $n\mathbf{Cat}$ admits h -pullbacks, is to show that the four-tuple $(\mathbb{P}, P, Q, \varepsilon)$ satisfies universal property of h -Pullbacks (UP 2.12). To this aim, let us suppose a n -category \mathbb{X} been given, together with morphisms and 2-morphisms

$$M : \mathbb{X} \rightarrow \mathbb{A}, \quad N : \mathbb{X} \rightarrow \mathbb{C}, \quad \omega : MF \Rightarrow NG$$

On objects, as \mathbb{P}_0 is a limit in \mathbf{Set} , it suffices to consider the cone over the same diagram defining the latter, whose commutativity is a consequence of the very definition of ω :



This yields a unique map $L_0 : \mathbb{X}_0 \rightarrow \mathbb{P}_0$, such that:

$$L_0 P_0 = M_0, \quad L_0 Q_0 = N_0, \quad L_0 \varepsilon_0 = \omega_0 \quad (3.18)$$

On homs, let us fix objects x_0 and x'_0 in \mathbb{P} . By the universal property in dimension $n - 1$, the four-tuple

$$(\mathbb{X}_1(x_0, x'_0), M_1^{x_0, x'_0}, N_1^{x_0, x'_0}, \varepsilon_1^{x_0, x'_0})$$

gives a unique morphism $L_1^{x_0, x'_0} : \mathbb{X}_1(x_0, x'_0) \rightarrow \mathbb{P}_1(Lx_0, Lx'_0)$ such that

$$\begin{aligned} L_1^{x_0, x'_0} \bullet^0 P_1^{Lx_0, Lx'_0} &= M_1^{x_0, x'_0} \\ L_1^{x_0, x'_0} \bullet^0 Q_1^{Lx_0, Lx'_0} &= N_1^{x_0, x'_0} \\ L_1^{x_0, x'_0} \bullet^0 \varepsilon_1^{Lx_0, Lx'_0} &= \omega_1^{x_0, x'_0} \end{aligned} \quad (3.19)$$

Claim: the pair $(L_0, L_1^{-, -})$ constitutes a n -functor $L : \mathbb{X} \rightarrow \mathbb{P}$.

Proof. The proof is divided in two parts.

1. *Functoriality w.r.t. compositions.*

Let us fix a triple x_0, x'_0, x''_0 of objects of \mathbb{X} . What we want to prove is the diagram below commutes:

$$\begin{array}{ccc} \mathbb{X}_1(x_0, x'_0) \times \mathbb{X}_1(x'_0, x''_0) & \xrightarrow{\mathbb{X}_\circ} & \mathbb{X}_1(x_0, x''_0) \\ L_1 \times L_1 \downarrow & & \downarrow L_1 \\ \mathbb{P}_1(Lx_0, Lx'_0) \times \mathbb{P}_1(Lx'_0, Lx''_0) & \xrightarrow[\mathbb{P}_\circ]{} & \mathbb{P}_1(Lx_0, Lx''_0) \end{array}$$

Then, let us consider the h -pullback defining $\mathbb{P}_1(Lx_0, Lx''_0)$. If we can show that the horizontal composition of both composites above with $\varepsilon_1^{p_0, p''_0}$ coincide, uniqueness forces $(L_1 \times L_1) \bullet^0 \mathbb{P}_\circ = \mathbb{X}_\circ \bullet^0 L_1$. Hence, let us follow the chain of equalities below:

$$\begin{array}{c} [x_0, x'_0] \times [x'_0, x''_0] \\ \downarrow \mathbb{X}_\circ \\ [x_0, x''_0] \\ \downarrow L_1 \\ [Lx_0, Lx''_0] \\ \downarrow \varepsilon_1^{Lx_0, Lx''_0} \\ [MFx_0, NGx''_0] \end{array} \stackrel{(1)}{=} \begin{array}{c} [x_0, x'_0] \times [x'_0, x''_0] \\ \downarrow \mathbb{X}_\circ \\ [x_0, x''_0] \\ \downarrow \omega_1^{x_0, x''_0} \\ [MFx_0, NGx''_0] \end{array} \stackrel{(2)}{=} \begin{array}{c} [x_0, x'_0] \times [x'_0, x''_0] \\ \swarrow \text{id} \times \omega_1^{x'_0, x''_0} \quad \searrow \omega_1^{x_0, x'_0} \times \text{id} \\ [x_0, x'_0] \times [MFx'_0, NGx''_0] \quad [MFx_0, NGx'_0] \times [x'_0, x''_0] \\ \searrow MF_1(-) \circ - \quad \swarrow - \circ NG_1(-) \\ [MFx_0, NGx''_0] \end{array}$$

(1) holds by the third equation of (3.19), (2) by functoriality w.r.t. composi-

tion of ω ,

$$\begin{array}{c}
 \begin{array}{ccc}
 & [x_0, x'_0] \times [x'_0, x''_0] & \\
 id \times L_1 \swarrow & & \searrow L_1 \times id \\
 [x_0, x'_0] \times [Lx'_0, Lx''_0] & & [Lx_0, Lx'_0] \times [x'_0, x''_0] \\
 \begin{array}{c} \text{---} id \times \varepsilon_1 \text{---} \\ \text{---} \varepsilon_1 \text{---} \end{array} & & \begin{array}{c} \text{---} \varepsilon_1 \text{---} \\ \text{---} Lx_0, Lx'_0 \times id \text{---} \end{array} \\
 \swarrow & & \searrow \\
 [x_0, x'_0] \times [LPFx'_0, LQGx''_0] & & [LPFx_0, LQGx'_0] \times [x'_0, x''_0] \\
 \swarrow LPF_1(-) \circ - & & \searrow - \circ LQG_1(-) \\
 & [MFx_0, NGx''_0] &
 \end{array}
 \end{array}
 \quad (3)$$

here, (3) is a full consequence of equations (3.19) and product interchange,

$$\begin{array}{ccc}
 \begin{array}{c}
 \begin{array}{ccc}
 & [x_0, x'_0] \times [x'_0, x''_0] & \\
 \downarrow L_1 \times L_1 & & \\
 [Lx_0, Lx'_0] \times [Lx'_0, Lx''_0] & & \\
 \begin{array}{c} \text{---} id \times \varepsilon_1 \text{---} \\ \text{---} \varepsilon_1 \text{---} \end{array} & & \begin{array}{c} \text{---} \varepsilon_1 \text{---} \\ \text{---} Lx_0, Lx'_0 \times id \text{---} \end{array} \\
 \swarrow & & \searrow \\
 [x_0, x'_0] \times [PFx'_0, QGx''_0] & & [PFx_0, QGx'_0] \times [x'_0, x''_0] \\
 \swarrow PF_1(-) \circ - & & \searrow - \circ QG_1(-) \\
 & [PFx_0, QGx''_0] &
 \end{array}
 \end{array}
 \quad (4)
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{ccc}
 & [x_0, x'_0] \times [x'_0, x''_0] & \\
 \downarrow L_1 \times L_1 & & \\
 [Lx_0, Lx'_0] \times [Lx'_0, Lx''_0] & & \\
 \downarrow \mathbb{P}_O & & \\
 [Lx_0, Lx''_0] & & \\
 \begin{array}{c} \text{---} \varepsilon_1 \text{---} \\ \text{---} Lx_0, Lx'_0 \text{---} \end{array} & & \\
 \downarrow & & \\
 [MFx_0, NGx''_0] & &
 \end{array}
 \end{array}
 \quad (5)$$

(4) is obtained sliding the L_1 's up, and functoriality w.r.t. composition of ε gives (5).

2. Functoriality w.r.t. units.

Let us fix an object x_0 in \mathbb{X} . What we want to prove is the diagram below commutes:

$$\begin{array}{ccc}
 \mathbb{I} & \xrightarrow{u(x_0)} & \mathbb{X}_1(x_0, x_0) \\
 & \searrow u(Lx_0) & \downarrow L_1 \\
 & & \mathbb{P}_1(Lx_0, Lx_0)
 \end{array}$$

We proceed in a similar way. Let us consider the pullback defining $\mathbb{P}_1(Lx_0, Lx''_0)$. If we show that the horizontal composition of both composites above with

$\varepsilon_1^{Lx_0, Lx_0}$ coincide, uniqueness forces $u(x_0) \bullet^0 L_1 = u(Lx_0)$. Hence, let us follow the chain of equalities below:

$$\begin{array}{c}
 \Downarrow \\
 \downarrow u(x_0) \\
 [x_0, x_0] \\
 \downarrow L_1 \\
 [Lx_0, Lx_0] \\
 \leftarrow \varepsilon_1^{Lx_0, Lx_0} \rightarrow \\
 \downarrow \\
 [MFx_0, NGx_0]
 \end{array}
 \stackrel{(1)}{=}
 \begin{array}{c}
 \Downarrow \\
 \downarrow u(x_0) \\
 [x_0, x_0] \\
 \leftarrow \omega_1^{Lx_0, Lx_0} \rightarrow \\
 \downarrow \\
 [MFx_0, NGx_0]
 \end{array}
 \stackrel{(2)}{=}
 \begin{array}{c}
 \Downarrow \\
 \leftarrow Id_{[\omega_{x_0}]} \rightarrow \\
 \downarrow \\
 [MFx_0, NGx_0]
 \end{array}
 \stackrel{(3)}{=}
 \begin{array}{c}
 \Downarrow \\
 \downarrow u(Lx_0) \\
 [Lx_0, Lx_0] \\
 \leftarrow \varepsilon_1^{Lx_0, Lx_0} \rightarrow \\
 \downarrow \\
 [MFx_0, NGx_0]
 \end{array}$$

(1) holds by the third of the (3.19), (2) is just ω , the functoriality of units, and since by (3.18), $\omega_{x_0} = \varepsilon_{Lx_0}$, (3) is obtained by ε unit functoriality. \square

Once we have verified that L is a n -functor, equations (3.18) and (3.19) taken together are exactly conditions 1., 2. and 3. of *Universal Property 2.12*.

What is still missing is uniqueness, but this is implied by the proofs. In fact let us suppose there is another $\hat{L}\mathbb{X} \rightarrow \mathbb{P}$ satisfying universal property. Conditions 1., 2. and 3. of (2.12) imply:

$$\hat{L}_0 P_0 = M_0, \quad \hat{L}_0 Q_0 = N_0, \quad \hat{L}_0 \varepsilon_0 = \omega_0$$

and

$$\begin{aligned}
 \hat{L}_1^{x_0, x'_0} \bullet^0 P_1^{\hat{L}x_0, \hat{L}x'_0} &= M_1^{x_0, x'_0} \\
 \hat{L}_1^{x_0, x'_0} \bullet^0 Q_1^{\hat{L}x_0, \hat{L}x'_0} &= N_1^{x_0, x'_0} \\
 \hat{L}_1^{x_0, x'_0} \bullet^0 \varepsilon_1^{\hat{L}x_0, \hat{L}x'_0} &= \omega_1^{x_0, x'_0}
 \end{aligned}$$

Since L_0 and the L_1 's were determined univocally by Universal Properties of limits in \mathbf{Set} and of h -pullbacks in $(n-1)\mathbf{Cat}$, uniqueness of those forces

$$\hat{L}_0 = L_0 \quad \text{and} \quad \hat{L}_1^{-, -} = L_1^{-, -}$$

Hence we proved the

Theorem 3.8. *The sesqui-category $n\mathbf{Cat}$ admits h -pullbacks.*

3.6.5 Pullbacks and h -pullbacks

It is possible to recover the usual notion of pullback by means of a similar inductive construction: a pullback is a universal triple $\langle \mathbb{Q}, P, Q \rangle$ such that $PF = QG$:

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{Q} & \mathbb{C} \\ P \downarrow & (pb) & \downarrow G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array}$$

where \mathbb{Q}_0 is the pullback in \mathbf{Set}

$$\begin{array}{ccc} \mathbb{Q}_0 & \xrightarrow{Q_0} & \mathbb{C}_0 \\ P_0 \downarrow & (pb) & \downarrow G_0 \\ \mathbb{A}_0 & \xrightarrow{F_0} & \mathbb{B}_0 \end{array}$$

and for every pair of objects (a_0, c_0) and (a'_0, c'_0) of \mathbb{Q}_0 , the following pullback in $(n-1)\mathbf{Cat}$

$$\begin{array}{ccc} \mathbb{Q}_1((a_0, c_0), (a'_0, c'_0)) & \xrightarrow{Q_1^{(a_0, c_0), (a'_0, c'_0)}} & \mathbb{C}_1(c_0, c'_0) \\ P_1^{(a_0, c_0), (a'_0, c'_0)} \downarrow & (pb) & \downarrow G_1^{c_0, c'_0} \\ \mathbb{A}_1(a_0, a'_0) & \xrightarrow{F_1^{a_0, a'_0}} & \mathbb{B}_1(Fa_0 = Gc_0, Fa'_0 = Gc'_0) \end{array}$$

In fact this gives an h -pullback in the trivial (= 2-discrete) sesqui-category over the category $[n\mathbf{Cat}]$: the triple $\langle \mathbb{Q}, P, Q \rangle$ such that $PF = QG$ may be seen as a four-tuple $\langle \mathbb{Q}, P, Q, id : PF \Rightarrow QG \rangle$, and so on...

Chapter 4

n -Groupoids and exact sequences

The sesqui-category $n\mathbf{Cat}$ of strict and small n -categories, defined so far, has a naturally arising notion of equivalence that may be defined recursively. This gives a notion of n -groupoid, equivalent to that of Kapranov and Voevodsky in [KV91], as a *weakly invertible strict n -category* (see Appendix A for a comparison).

4.1 n -Equivalences

Definition 4.1. Let n -category morphism $F : \mathbb{C} \rightarrow \mathbb{D}$ be given. F is called equivalence of n -categories if it satisfies the following properties:

$$\boxed{n = 0}$$

F is an isomorphism in \mathbf{Set} .

$$\boxed{n > 0}$$

1. F is essentially surjective on objects, i.e. for every object d_0 of \mathbb{D} , there exists an object c_0 of \mathbb{C} and a 1-cell

$d_1 : Fc_0 \rightarrow d_0$ such that for every d'_0 in \mathbb{C} , the morphisms

$$\begin{aligned} d_1 \circ - & : \mathbb{D}_1(d_0, d'_0) \rightarrow \mathbb{D}_1(Fc_0, d'_0) \\ - \circ d_1 & : \mathbb{D}_1(d'_0, Fc_0) \rightarrow \mathbb{D}_1(d'_0, d_0) \end{aligned}$$

are equivalences of $(n-1)$ categories.

2. for every pair c_0, c'_0 in \mathbb{C} ,

$$F_1^{c_0, c'_0} : \mathbb{C}_1(c_0, c'_0) \rightarrow \mathbb{D}_1(Fc_0, Fc'_0)$$

is an equivalence of $(n-1)$ categories.

From definition above, one gets the following

Definition 4.2. A 1-cell $c_1 : c_0 \rightarrow c'_0$ of a n -category \mathbb{C} is said to be weakly invertible, or simply an equivalence, if, for every object \bar{c}_0 of \mathbb{C} , the morphisms

$$\begin{aligned} c_1 \circ - & : \mathbb{C}_1(c'_0, \bar{c}_0) \rightarrow \mathbb{C}_1(c_0, \bar{c}_0) \\ - \circ c_1 & : \mathbb{C}_1(\bar{c}_0, c_0) \rightarrow \mathbb{C}_1(\bar{c}_0, c'_0) \end{aligned}$$

are (natural) equivalences of $(n-1)$ categories.

4.1.1 Inverses

When a 1-cell is weakly invertible, then it has indeed left and right (quasi) inverses. In fact for $c_1 : c_0 \rightarrow c'_0$,

$$c_1 \circ - : \mathbb{C}_1(c'_0, c_0) \rightarrow \mathbb{C}_1(c_0, c_0)$$

to be an equivalence implies that for the 1-cell $1_{c_0} : c_0 \rightarrow c_0$ there exists a pair

$$(c_1^*, c_2 : c_1 \circ c_1^* \xrightarrow{\sim} 1_{c_1}),$$

similarly for

$$- \circ c_1 : \mathbb{C}_1(c'_0, c_0) \rightarrow \mathbb{C}_1(c'_0, c'_0)$$

implies there exists a pair

$$(c_1^\dagger, c'_2 : c_1^\dagger \circ c_1 \xrightarrow{\sim} 1_{c'_1}).$$

4.1.2 Properties

Lemma 4.3. Let n -functors $\mathbb{C} \xrightarrow{F} \mathbb{D} \xrightarrow{G} \mathbb{E}$ be given. Then if F and G are equivalences, so is $F \bullet^0 G$.

Proof. The composite of isomorphisms in Set is trivially an isomorphism. Hence we may suppose $n > 0$.

1. for every pair c_0, c'_0 , $[F \bullet^0 G]_1^{c_0, c'_0}$ is an equivalence. In fact

$$[F \bullet^0 G]_1^{c_0, c'_0} = F_1^{c_0, c'_0} \bullet^0 G_1^{F c_0, F c'_0}$$

two component on the right-hand side are indeed equivalences by hypothesis, and so is their composites by induction.

2. For any object e_0 of \mathbb{E} there exists a pair $(d_0, e_1 : Gd_0 \rightarrow e_0)$. Similarly, for any object d_0 in \mathbb{D} there is a pair $(c_0, d_1 : Fc_0 \rightarrow d_0)$. Hence, for given e_0 , those produce a pair

$$(c_0, G(Fc_0) \xrightarrow{Gd_1} Gd_0 \xrightarrow{e_1} e_0)$$

That left and right 0-compositions (in \mathbb{D}) with $Gd_1 \circ e_1$ are equivalences, is a statement involving a composition of equivalences of $(n-1)$ categories, hence given by induction. In fact, by definition

$$- \circ (Gd_1 \circ e_1) = (- \circ Gd_1) \bullet^0 (- \circ e_1)$$

and

$$(Gd_1 \circ e_1) \circ - = (e_1 \circ -) \bullet^0 (Gd_1 \circ -)$$

□

Definition 4.4. A 2-morphism of n -categories $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$ is an equivalence 2-morphism, or n -natural equivalence, if

$$\boxed{n = 1}$$

α is a natural isomorphism.

$$\boxed{n > 1}$$

1. for any object c_0 in \mathbb{C} , the 1-cell α_{c_0} is an equivalence
2. for any pair of objects c_0, c'_0 in \mathbb{C} , the $(n-1)$ transformation $\alpha_1^{c_0, c'_0}$ is an equivalence 2-morphism

It is not precisely in the aims of this work, nevertheless it is worth mentioning the following

Proposition 4.5. n -categories, n -functors and n -natural equivalences form a sesqui-category, denoted $n\mathbf{Cat}_{eq}$,

Notice that in $n\mathbf{Cat}_{eq}$, and *a fortiori* in $n\mathbf{Gpd}$ (t.b.d.) equivalences have more nice properties, like they are h -pullback stable, have the 2of3 property and so on.

Definition 4.6. A morphism of n -categories $F : \mathbb{C} \rightarrow \mathbb{D}$ is called h -surjective if

$$\boxed{n = 0}$$

F is a surjective map.

$n > 0$

1. F is essentially surjective on objects, i.e.

for every object d_0 of \mathbb{D} , there exists a pair $(c_0, d_1 : Lc_0 \xrightarrow{\sim} d_0)$, with d_1 an equivalence,

2. for every pair of objects c_0, c'_0 of \mathbb{C} , the morphism

$$F_1^{c_0, c'_0} : \mathbb{C}_1(c_0, c'_0) \rightarrow \mathbb{D}_1(Fc_0, Fc'_0)$$

is h -surjective.

Definition 4.7. A morphism of n -categories $F : \mathbb{C} \rightarrow \mathbb{D}$ is called faithful if

$n = 0$

F is an injective map.

$n > 0$

for every pair c_0, c'_0 , the $(n-1)$ -functor $F_1^{c_0, c'_0}$ is faithful.

The notion of h -surjective is weaker than (implied by) that of equivalence. In fact, more is true:

Proposition 4.8. A morphism of n -categories $F : \mathbb{C} \rightarrow \mathbb{D}$ is an equivalence precisely when it is faithful and h -surjective.

Proof. When $n = 0$ this is the characterization of bijective maps as injective plus surjective. Hence suppose $n > 0$.

Let F be an equivalence. Then F is essentially surjective by definition. More, for every c_0, c'_0 , $F_1^{c_0, c'_0}$ is an equivalence in $(n-1)\mathbf{Cat}$, therefore h -surjective by induction. Finally, the last is also faithful by induction, and this concludes the first implication.

Conversely, let F be faithful and h -surjective. Then it is essentially surjective by definition. More, for every c_0, c'_0 , $F_1^{c_0, c'_0}$'s are faithful and h -surjective, and inductive hypothesis implies they are equivalences. \square

Notice that faithfulness can be reformulated saying that the n -functor is *surjective on equations*. In fact, from the globular point of view, this is equivalent to saying that equal n -cells in the image of F come from equal n -cells of \mathbb{C} . Under this perspective, to be h -surjective amounts to being (weakly) surjective in any dimension, up to $n-1$, and last proposition says precisely that an equivalence is (weakly) surjective on k -cell, with $0 \leq k \leq n$.

Remark 4.9. The notions of *h*-surjective morphisms and of equivalences reduce to well known ones, when considered in low dimension. In fact, in dimension one those are stated explicitly in the definitions as the first step of the inductive process. For $n = 1$ an *h*-surjective morphism is a functor which is full and essentially surjective on object. Hence the notion of equivalence is the usual one.

Finally we state a useful Lemma, whose proof is part of the proof of *Lemma 4.3*:

Lemma 4.10. *Let n -functors $\mathbb{C} \xrightarrow{F} \mathbb{D} \xrightarrow{G} \mathbb{E}$ be given. Then if F and G are *h*-surjective, so is $F \bullet^0 G$.*

4.2 *n*-Groupoids

Definition 4.11. *The definition is inductive on n .*

$n = 0$

A 0-groupoid is a 0-category, i.e. a set.

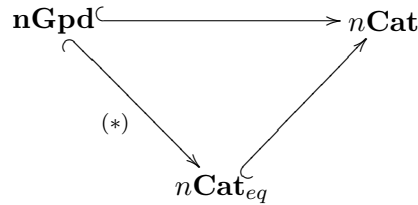
$n > 0$

A *n*-groupoid is a *n*-category \mathbb{C} such that:

1. every 1-cell of \mathbb{C} is an equivalence;
2. for every pair of objects c_0, c'_0 of \mathbb{C} the $(n - 1)$ category $\mathbb{C}_1(c_0, c'_0)$ is a $(n - 1)$ groupoid.

We denote by $n\mathbf{Gpd}$ the sub-sesqui-category of $n\mathbf{Cat}$ generated by *n*-groupoids.

Proposition 4.12. *For every given natural number n , the following is a diagram of inclusions:*



Proof. The case $= 1$ is well known, hence suppose $n > 1$.

The only inclusion to be proved is the one marked $(*)$. To this end, it suffice to show that 2-morphisms of *n*-groupoids are equivalences. But, condition

1. of *Lemma 4.4* above is automatically satisfied, as n -groupoid's 1-cells are always equivalences, condition 2. is given by induction. \square

Proposition 4.13. *$n\mathbf{Gpd}$ is closed under h -pullbacks.*

Proof. For $n = 0$ the result holds trivially. Hence let us suppose $n > 0$. For the h -pullback

$$\langle \mathbb{P}, P, Q, \varepsilon \rangle$$

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{Q} & \mathbb{C} \\ P \downarrow & \nearrow \varepsilon & \downarrow G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array}$$

a 1-cell

$$\begin{array}{ccccccc} (a_0 & , & Fa_0 & \xrightarrow{b_1} & Gc_0 & , & c_0) \\ a_1 \downarrow & & Fa_1 \downarrow & \nearrow b_2 & \downarrow Gc_1 & & \downarrow c_1 \\ (a'_0 & , & Fa'_0 & \xrightarrow{b'_1} & Gc'_0 & , & c'_0) \end{array}$$

induces equivalences because a_1, c_1 and b_2 do. Turning to homs, the statement is relative to $(n - 1)$ groupoids and holds by induction. \square

4.3 The sesqui-functor π_0

Purpose of this section is to introduce the family of sesqui-functors $\{\pi_0^{(n)}\}_{n \in \mathbb{N}^*}$ that extends the iso-classes functor $\mathbf{Gpd} \rightarrow \mathbf{Set}$.

Definition/Proposition 4.14. *For any integer $n > 0$, there exists a classifying sesqui-functor*

$$\pi_0^{(n)} : n\mathbf{Gpd} \rightarrow (n - 1)\mathbf{Gpd}$$

according to the following inductive definition.

Moreover, it commutes with finite products and it preserves equivalences.

$$\boxed{n = 1}$$

$$\pi_0^{(1)} : n\mathbf{Gpd} \rightarrow (n - 1)\mathbf{Gpd}$$

is the functor (= trivial sesqui-functor) $\mathbf{Gpd} \rightarrow \mathbf{Set}$ that assigns to a groupoid \mathbb{C} the set $|\mathbb{C}|$ of isomorphism classes of objects of \mathbb{C} .

It commutes with finite products: in fact the terminal set $\{*\}$ is exactly the classified terminal groupoid \mathbb{I} , and in a product of groupoids $\mathbb{C} \times \mathbb{D}$, an isomorphism (c_1, d_1) is a pair of isomorphisms c_1 in \mathbb{C} and d_1 in \mathbb{D} .

Finally, it sends equivalences of category in isomorphism-maps.

$$\boxed{n > 1}$$

4.3.1 π_0 on objects

Let a n -groupoid \mathbb{C} be given. Then, $\pi_0^{(n)}\mathbb{C} = ([\pi_0^{(n)}\mathbb{C}]_0, [\pi_0^{(n)}\mathbb{C}]_1(-, -))$, where

- $[\pi_0^{(n)}\mathbb{C}]_0 = \mathbb{C}_0$
- for every pair c_0, c'_0 in $[\pi_0^{(n)}\mathbb{C}]_0$,

$$[\pi_0^{(n)}\mathbb{C}]_1(c_0, c'_0) = \pi_0^{(n-1)}(\mathbb{C}_1(c_0, c'_0))$$

For a triple of objects c_0, c'_0, c''_0 , composition is the dotted arrow below:

$$\begin{array}{ccc} [\pi_0^{(n)}\mathbb{C}]_1(c_0, c'_0) \times [\pi_0^{(n)}\mathbb{C}]_1(c'_0, c''_0) & \xrightarrow{\pi_0^{(n)}\mathbb{C}_\circ} & [\pi_0^{(n)}\mathbb{C}]_1(c_0, c''_0) \\ \text{\scriptsize def} \parallel & & \parallel \\ \pi_0^{(n-1)}(\mathbb{C}_1(c_0, c'_0)) \times \pi_0^{(n-1)}(\mathbb{C}_1(c'_0, c''_0)) & & \pi_0^{(n-1)}(\mathbb{C}_1(c_0, c''_0)) \\ \text{\scriptsize (1)} \parallel & & \parallel \text{\scriptsize def} \\ \pi_0^{(n-1)}(\mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c''_0)) & \xrightarrow{\pi_0^{(n-1)}(\mathbb{C}_\circ)} & \pi_0^{(n-1)}(\mathbb{C}_1(c_0, c''_0)) \end{array}$$

For an object c_0 , unit morphism is the dotted arrow below:

$$\begin{array}{ccc} \mathbb{I}_{(n)} & \xrightarrow{u(c_0)} & [\pi_0^{(n)}\mathbb{C}]_1(c_0, c_0) \\ \text{\scriptsize (2)} \parallel & & \parallel \text{\scriptsize def} \\ \pi_0^{(n-1)}(\mathbb{I}_{(n-1)}) & \xrightarrow{\pi_0^{(n-1)}(u(c_0))} & \pi_0^{(n-1)}(\mathbb{C}_1(c_0, c_0)) \end{array}$$

Equalities (1) and (2) hold because $\pi_0^{(n-1)}$ commutes with (finite) products by induction hypothesis.

Claim 4.15. *The pair $([\pi_0^{(n)}\mathbb{C}]_0, [\pi_0^{(n)}\mathbb{C}]_1^-, -)$, with composition and units as defined above, satisfies axioms for a $(n-1)$ category.*

Moreover, $\pi_0^{(n)}\mathbb{C}$ is a $(n-1)$ groupoid.

Proof. We have to prove associativity and unit axioms.

Concerning associativity, let objects c_0, c'_0, c''_0, c'''_0 be given. Let us consider the following equalities of morphisms

$$[\pi_0^{(n)}\mathbb{C}]_1(c_0, c'_0) \times [\pi_0^{(n)}\mathbb{C}]_1(c'_0, c''_0) \times [\pi_0^{(n)}\mathbb{C}]_1(c''_0, c'''_0) \rightarrow [\pi_0^{(n)}\mathbb{C}]_1(c_0, c'''_0),$$

$$\begin{aligned} (\pi_0\mathbb{C} \circ_{c_0, c'_0, c''_0} id) \bullet^0 \pi_0\mathbb{C} \circ_{c'_0, c''_0, c'''_0} & \stackrel{(def)}{=} \pi_0^{(n-1)}(\mathbb{C} \circ_{c_0, c'_0, c''_0} id) \bullet^0 \pi_0^{(n-1)}(\mathbb{C} \circ_{c'_0, c''_0, c'''_0}) \\ & \stackrel{(1)}{=} \pi_0^{(n-1)}((\mathbb{C} \circ_{c_0, c'_0, c''_0} id) \bullet^0 \mathbb{C} \circ_{c'_0, c''_0, c'''_0}) \\ & \stackrel{(2)}{=} \pi_0^{(n-1)}((id \times \mathbb{C} \circ_{c'_0, c''_0, c'''_0}) \bullet^0 \mathbb{C} \circ_{c_0, c'_0, c''_0}) \\ & \stackrel{(3)}{=} \pi_0^{(n-1)}(id \times \mathbb{C} \circ_{c'_0, c''_0, c'''_0}) \bullet^0 \pi_0^{(n-1)}(\mathbb{C} \circ_{c_0, c'_0, c''_0}) \\ & \stackrel{(def)}{=} (id \times \pi_0\mathbb{C} \circ_{c'_0, c''_0, c'''_0}) \bullet^0 \pi_0\mathbb{C} \circ_{c_0, c'_0, c''_0} \end{aligned}$$

where, (1) and (3) hold by functoriality of $\pi_0^{(n-1)}$, (2) by associativity of 0-composition in \mathbb{C} .

Turning to left-units, for every pair c_0, c'_0 , one has the following equalities of morphisms:

$$[\pi_0^{(n)}\mathbb{C}]_1(c'_0, c_0) \rightarrow [\pi_0^{(n)}\mathbb{C}]_1(c'_0, c_0)$$

$$\begin{aligned} \lambda \bullet^0 (id \times u(c_0)) \bullet^0 \circ_{c'_0, c_0, c_0} & \stackrel{(def)}{=} \pi_0^{(n-1)}(\lambda) \bullet^0 \pi_0^{(n-1)}(id \times u(c_0)) \bullet^0 \pi_0^{(n-1)}(\circ_{c'_0, c_0, c_0}) \\ & \stackrel{(1)}{=} \pi_0^{(n-1)}(\lambda \bullet^0 (id \times u(c_0)) \bullet^0 \circ_{c'_0, c_0, c_0}) \\ & \stackrel{(2)}{=} \pi_0^{(n-1)}(id_{\mathbb{C}_1(c'_0, c_0)}) \\ & \stackrel{(3)}{=} id_{\pi_0^{(n-1)}\mathbb{C}_1(c'_0, c_0)} \\ & \stackrel{(def)}{=} id_{[\pi_0^{(n)}\mathbb{C}]_1(c'_0, c_0)} \end{aligned}$$

where 1 and 3 hold by functoriality of $\pi_0^{(n-1)}$, (2) by neutrality of 0-identities in \mathbb{C} .

Right units are dealt the same way.

Finally, in order to show that $\pi_0^{(n)}\mathbb{C}$ is a $(n-1)$ groupoid, two facts have to be proved:

1. all 1-cells of $\pi_0^{(n)}\mathbb{C}$ are equivalences.
2. all homs $[\pi_0^{(n)}\mathbb{C}]_1(c_0, c'_0)$ are $(n-2)$ groupoids.

The first fact is an easy consequence of the very definition of π_0 on compositions. In fact, let object \bar{c}_0 and 1-cell $\tilde{x} : c_0 \rightarrow c'_0$ in $\pi_0^{(n)}\mathbb{C}$ be given. Then by definition, $- \circ \tilde{x} = \pi_0^{(n-1)}(- \circ x)$, where $x = \tilde{x}$ if $n > 1$, or $\{x\}_\sim = \tilde{x}$ if $n = 1$. The result follows, for $\pi_0^{(n-1)}$ preserves equivalences.

To prove the second statement, let us consider the $(n-2)$ category $[\pi_0^{(n)}\mathbb{C}]_1(c_0, c'_0)$. This is defined to be $\pi_0^{(n-1)}(\mathbb{C}_1(c_0, c'_0))$, where $\mathbb{C}_1(c_0, c'_0)$ is a $(n-1)$ groupoid. For $\pi_0^{(n-1)}$ the result follows by induction. \square

4.3.2 π_0 on morphisms

Let a n -functor $F : \mathbb{C} \rightarrow \mathbb{D}$ be given. Then, $\pi_0^{(n)}F = ([\pi_0^{(n)}F]_0, [\pi_0^{(n)}F]_1^-, -)$, where

- $[\pi_0^{(n)}F]_0 = F_0$
- for every pair c_0, c'_0 in $[\pi_0^{(n)}\mathbb{C}]_0$

$$[\pi_0^{(n)}F]_1^{c_0, c'_0} = \pi_0^{(n-1)}(F_1^{c_0, c'_0})$$

Claim 4.16. *The pair $([\pi_0^{(n)}F]_0, [\pi_0^{(n)}F]_1^-, -)$ satisfies axioms for $(n-1)$ functors.*

Proof. We have to prove functoriality w.r.t. composition and units.

Concerning composition, let objects c_0, c'_0, c''_0 be given. Let us consider the following equalities of morphisms

$$\begin{aligned} [\pi_0^{(n)}\mathbb{C}]_1(c_0, c'_0) \times [\pi_0^{(n)}\mathbb{C}]_1(c'_0, c''_0) &\rightarrow [\pi_0^{(n)}\mathbb{D}]_1(Fc_0, Fc''_0) \\ \pi_0^{(n)}\mathbb{C}_0 \bullet^0 [\pi_0^{(n)}F]_1^{c_0, c'_0} &\stackrel{(def)}{=} \pi_0^{(n-1)}(\mathbb{C}_0) \bullet^0 \pi_0^{n-1}(F_1^{c_0, c'_0}) \\ &\stackrel{(1)}{=} \pi_0^{(n-1)}(\mathbb{C}_0 \bullet^0 F_1^{c_0, c'_0}) \\ &\stackrel{(2)}{=} \pi_0^{(n-1)}((F_1^{c_0, c'_0} \times F_1^{c'_0, c''_0}) \bullet^0 \mathbb{D}_0) \\ &\stackrel{(3)}{=} \pi_0^{(n-1)}(F_1^{c_0, c'_0} \times F_1^{c'_0, c''_0}) \bullet^0 \pi_0^{(n-1)}(\mathbb{D}_0) \\ &\stackrel{(def)}{=} ([\pi_0^{(n)}F]_1^{c_0, c'_0} \times [\pi_0^{(n)}F]_1^{c'_0, c''_0}) \bullet^0 \pi_0^{(n)}\mathbb{D}_0 \end{aligned}$$

(1) and (3) are justified by the sesqui-functor $\pi_0^{(n-1)}$ preserving composition, (2) is functoriality w.r.t. composition of F .

Turning to identities, for every c_0 one has the following equalities of morphisms

$$\begin{aligned} \mathbb{I}_{(n)} &\rightarrow [\pi_0^{(n)}]_1(Fc_0, Fc_0) \\ u(c_0) \bullet^0 [\pi_0^{(n)}\mathbb{D}]_1^{c_0, c_0} &\stackrel{(def)}{=} \pi_0^{(n-1)}(u(c_0)) \bullet^0 \pi_0^{(n-1)}(F_1^{c_0, c_0}) \\ &\stackrel{(1)}{=} \pi_0^{(n-1)}(u(c_0) \bullet^0 F_1^{c_0, c_0}) \\ &\stackrel{(2)}{=} \pi_0^{(n-1)}(u(Fc_0)) \\ &\stackrel{(def)}{=} u(Fc_0) \end{aligned}$$

where (1) holds for $\pi_0^{(n-1)}$ preserving composition, (2) by functoriality w.r.t. units of F . \square

4.3.3 The underlying functor

In order to be a sesqui-functor, π_0 must restrict to a functor between the underlying categories:

$$[\pi_0^{(n)}] : [\mathbf{nGpd}] \rightarrow [(\mathbf{n} - 1)\mathbf{Gpd}]$$

Claim 4.17. *Given the situation*

$$\mathbb{C} \xrightarrow{F} \mathbb{D} \xrightarrow{G} \mathbb{E}$$

in \mathbf{nGpd} , the assignments given above satisfy

1. $\pi_0^{(n)}(F \bullet^0 G) = \pi_0^{(n)} F \bullet^0 \pi_0^{(n)} G$
2. $\pi_0^{(n)}(id_{\mathbb{C}}) = id_{\pi_0^{(n)} \mathbb{C}}$

Proof. Let us check first number 1.

$$\begin{aligned} [\pi_0^{(n)}(F \bullet^0 G)]_0 &\stackrel{(def)}{=} [F \bullet^0 G]_0 \\ &\stackrel{(1)}{=} F_0 G_0 \\ &\stackrel{(def)}{=} [\pi_0^{(n)} F]_0 \bullet^0 [\pi_0^{(n)} G]_0 \end{aligned}$$

where (1) holds by composition of n-functors.

Moreover, for objects c_0, c'_0 ,

$$\begin{aligned} [\pi_0^{(n)}(F \bullet^0 G)]_1^{c_0, c'_0} &\stackrel{(def)}{=} \pi_0^{(n-1)}([F \bullet^0 G]_1^{c_0, c'_0}) \\ &\stackrel{(1)}{=} \pi_0^{(n-1)}(F_1^{c_0, c'_0} \bullet^0 G_1^{F c_0, F c'_0}) \\ &\stackrel{(2)}{=} \pi_0^{(n-1)}(F_1^{c_0, c'_0}) \bullet^0 \pi_0^{(n-1)}(G_1^{F c_0, F c'_0}) \\ &\stackrel{(def)}{=} \pi_0^{(n)}(F_1^{c_0, c'_0}) \bullet^0 \pi_0^{(n)}(G_1^{F c_0, F c'_0}) \end{aligned}$$

where (1) holds again by composition of n-functors, and (2) by functoriality of $\pi_0^{(n-1)}$.

Now, let us check number 2.

$$\begin{aligned} [\pi_0^{(n)}(id_{\mathbb{C}})]_0 &= [id_{\mathbb{C}}]_0 \\ &= id_{\mathbb{C}_0} \\ &= id_{[\pi_0^{(n)} \mathbb{C}]_0} \\ &= [id_{\pi_0^{(n)} \mathbb{C}}]_0 \end{aligned}$$

follows straight from definitions.

For objects c_0, c'_0

$$\begin{aligned}
 [\pi^{(n)}(id_{\mathbb{C}})]_1^{c_0, c'_0} &= \pi_0^{(n-1)}([id_{\mathbb{C}}]_1^{c_0, c'_0}) \\
 &= \pi_0^{(n-1)}(id_{\mathbb{C}_1(c_0, c'_0)}) \\
 &\stackrel{(1)}{=} id_{\pi_0^{(n-1)}(\mathbb{C}_1(c_0, c'_0))} \\
 &= id_{[\pi_0^{(n)}\mathbb{C}]_1(c_0, c'_0)} \\
 &= [id_{\pi_0^{(n)}\mathbb{C}}]_1^{c_0, c'_0}
 \end{aligned}$$

where every equality above comes from definitions, but (1) that holds by functoriality of $\pi_0^{(n-1)}$. \square

4.3.4 π_0 on 2-morphisms

The *action* of $\pi_0^{(n)}$ on 2-morphisms is more sensible to define. Indeed, on objects and morphisms we had the *object-part* of definitions that were mere equalities, while the homs were given by induction. Now the situation is different, since the *object-part* of a 2-cell is a map involving also the 1-cells of codomain n -groupoid. That is the reason why we must analyze carefully what happens in low dimension, in order to start induction properly.

$$\boxed{n = 2}$$

Let us consider $\pi_0'' : 2\mathbf{Gpd} \rightarrow \mathbf{Gpd}$:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & F & \\
 \mathbb{C} & \begin{array}{c} \Downarrow \alpha \\ \Downarrow \end{array} & \mathbb{D} \\
 & G &
 \end{array} & \mapsto & \begin{array}{ccc}
 & \pi_0'' F & \\
 \pi_0'' \mathbb{C} & \begin{array}{c} \Downarrow \pi_0'' \alpha \\ \Downarrow \end{array} & \pi_0'' \mathbb{D} \\
 & \pi_0'' G &
 \end{array}
 \end{array}$$

(in order to simplify notation, for low dimensions we use primes).

Now it is clear that $[\pi_0''\mathbb{D}]_1 = \mathbb{D}_1 / \sim$, where \sim is the equivalence relation on the set of 1-cells of \mathbb{D} given by iso-2-cells. Call $p : \mathbb{D}_1 \rightarrow \mathbb{D}_1 / \sim$ the canonic projection onto the quotient. Then we let

$$[\pi_0''\alpha]_0 = \alpha_0 \cdot p$$

This is well defined, since equivalence classes in \mathbb{D}_1 respect 1-cell's sources and targets. Moreover they are compatible with 0-composition, *i.e.* $p(d_1 \circ d'_1) = p(d_1) \circ p(d'_1)$. In fact for every object c_0 of \mathbb{C} , one has

$$[\pi_0''\alpha]_0(c_0) = \{\alpha_0(c_0)\}_{\sim} : Fc_0 = [\pi_0''F](c_0) \rightarrow [\pi_0''G](c_0) = Gc_0$$

Hence if we choose a 1-cell $\tilde{c}_1 : c_0 \rightarrow c'_0$, say $\tilde{c}_1 = \{c_1\}_\sim$, then π_0'' sends the 2-isomorphism $\alpha_1^{c_0, c'_0}(c_1)$ in the equality

$$[\pi_0''\alpha]_{c_0} \circ \{Gc_1\}_\sim = \{Fc_1\}_\sim \circ [\pi_0''\alpha]_{c'_0}$$

This proves that $\pi_0''\alpha$ is a natural isomorphism of groupoids, i.e. a 2-morphism in **Gpd**.

$n > 2$

More generally, suppose we are given a 2-morphism $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$ in $n\mathbf{Gpd}$. Then we define $\pi_0^{(n)}\alpha$ as the pair $([\pi_0^{(n)}\alpha]_0, [\pi_0^{(n)}\alpha]_1^{-, -})$, where

- $[\pi_0^{(n)}\alpha]_0 = \alpha_0$,
- for every pair of objects c_0, c'_0 of \mathbb{C} ,

$$[\pi_0^{(n)}\alpha]_1^{c_0, c'_0} = \pi_0^{(n-1)}(\alpha_1^{c_0, c'_0}).$$

Claim 4.18. *The pair $([\pi_0^{(n)}\alpha]_0, [\pi_0^{(n)}\alpha]_1^{-, -})$ satisfies axioms for $(n-1)$ transformations.*

Proof. It is well-defined on objects, since $n > 2$. Moreover, it is well-defined also on homs. In fact, given objects c_0 and c'_0 , consider the diagram:

$$\begin{array}{ccccc}
 [\pi_0^{(n)}\mathbb{C}]_1(c_0, c'_0) & \xrightarrow{[\pi_0^{(n)}G]_1^{c_0, c'_0}} & & & [\pi_0^{(n)}\mathbb{D}]_1(Gc_0, Gc'_0) \\
 \downarrow [\pi_0^{(n)}F]_1^{c_0, c'_0} & \searrow & \pi_0^{(n-1)}(\mathbb{C}_1(c_0, c'_0)) \xrightarrow{\pi_0^{(n-1)}G_1^{c_0, c'_0}} \pi_0^{(n-1)}(\mathbb{D}_1(Gc_0, Gc'_0)) & \nearrow & \downarrow [\pi_0^{(n)}\alpha]_{c_0} \circ - \\
 & & \downarrow \pi_0^{(n-1)}F_1^{c_0, c'_0} & \swarrow \pi_0^{(n-1)}\alpha_1^{c_0, c'_0} & \downarrow \pi_0^{(n-1)}(\alpha_{c_0} \circ -) \\
 & & \pi_0^{(n-1)}(\mathbb{D}_1(Fc_0, Fc'_0)) \xrightarrow{\pi_0^{(n-1)}(-\circ\alpha_{c'_0})} \pi_0^{(n-1)}(\mathbb{D}_1(Fc_0, Gc'_0)) & & \\
 \downarrow & \nearrow & & \searrow & \downarrow [\pi_0^{(n)}\alpha]_{c'_0} \circ - \\
 [\pi_0^{(n)}\mathbb{D}]_1(Fc_0, Fc'_0) & \xrightarrow{-\circ[\pi_0^{(n)}\alpha]_{c'_0}} & & & [\pi_0^{(n)}\mathbb{D}]_1(Gc_0, Gc'_0)
 \end{array}$$

Up and left squares are the definition of $\pi_0^{(n)}$ on n-functors (w.r.t. hom-components), down and right commute by definition of composition in $\pi_0^{(n)}(\mathbb{D})$, and of $[\pi_0^{(n)}\alpha]_0$ above.

And that is all. In fact, coherence w.r.t. composition and units are satisfied because their diagrams-equations are π_0^{n-1} of corresponding diagrams-equations that hold for α . \square

For the same reason $\pi_0^{(n)}$

1. is functorial w.r.t. vertical composition and units of 2-morphisms
2. preserves reduced horizontal composition

i.e. it is a sesqui-functor.

Proof. 1. Suppose we are given

$$\omega : E \Rightarrow F : \mathbb{C} \rightarrow \mathbb{D} \quad \alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$$

in $n\mathbf{Gpd}$. When $n > 2$, for every object c_0 in $\pi_0^{(n)}\mathbb{C}$ one has

$$\begin{aligned} [\pi_0^{(n)}(\omega\alpha)]_0(c_0) &\stackrel{(1)}{=} (\omega\alpha)_0(c_0) \\ &\stackrel{(2)}{=} \omega_0 c_0 \circ \alpha_0 c_0 \\ &\stackrel{(3)}{=} [\pi_0^{(n)}(\omega)]_0(c_0) \circ [\pi_0^{(n)}(\alpha)]_0(c_0) \end{aligned}$$

where (1) and (3) hold by definition of $\pi_0^{(n)}$, and (2) by definition of vertical composition. When $n = 2$, one has

$$\begin{aligned} [\pi_0^{(n)}(\omega\alpha)]_0(c_0) &\stackrel{(1)}{=} p((\omega\alpha)_0(c_0)) \\ &\stackrel{(2)}{=} p(\omega_0 c_0 \circ \alpha_0 c_0) \\ &\stackrel{(3)}{=} p(\omega_0 c_0) \circ p(\alpha_0 c_0) \\ &\stackrel{(4)}{=} [\pi_0^{(n)}(\omega)]_0(c_0) \circ [\pi_0^{(n)}(\alpha)]_0(c_0) \end{aligned}$$

where (1) and (4) hold by definition of $\pi_0^{(n)}$, (2) by definition of vertical composition and (3) for p is compatible with 0-composition.

Now suppose we are given objects c_0, c'_0 of $\pi_0^{(n)}\mathbb{C}$. Then one has:

$$\begin{aligned} [\pi_0^{(n)}(\omega\alpha)]_1^{c_0, c'_0} &\stackrel{(1)}{=} \pi_0^{(n-1)}([\omega\alpha]_1^{c_0, c'_0}) \\ &\stackrel{(2)}{=} \pi_0^{(n-1)}((\alpha_1^{c_0, c'_0} \bullet_R^0 [\omega_{c_0} \circ -]) \bullet^1 (\omega_1^{c_0, c'_0} \bullet_R^0 [- \circ \alpha_{c'_0}])) \\ &\stackrel{(3)}{=} (\pi_0^{(n-1)}\alpha_1^{c_0, c'_0} \bullet_R^0 \pi_0^{(n-1)}[\omega_{c_0} \circ -]) \bullet^1 (\pi_0^{(n-1)}\omega_1^{c_0, c'_0} \bullet_R^0 \pi_0^{(n-1)}[- \circ \alpha_{c'_0}]) \\ &\stackrel{(4)}{=} ([\pi_0^{(n)}\alpha]_1^{c_0, c'_0} \bullet_R^0 \pi_0^{(n-1)}[\omega_{c_0} \circ -]) \bullet^1 ([\pi_0^{(n)}\omega]_1^{c_0, c'_0} \bullet_R^0 \pi_0^{(n-1)}[- \circ \alpha_{c'_0}]) \\ &\stackrel{(5)}{=} ([\pi_0^{(n)}\alpha]_1^{c_0, c'_0} \bullet_R^0 [[\pi_0^{(n)}\omega]_{c_0} \circ -]) \bullet^1 ([\pi_0^{(n)}\omega]_1^{c_0, c'_0} \bullet_R^0 [- \circ [\pi_0^{(n)}\alpha]_{c'_0}]) \end{aligned}$$

where (1) and (4) hold by the definition of $\pi_0^{(n)}$ on 2-morphisms w.r.t. homs, (2) by definition of vertical composition, (3) because $\pi_0^{(n)}$ is a sesqui-functor (induction), (5) by definition of $\pi_0^{(n)}$ 0-composition.

Concerning units, let us consider $id_F : F \Rightarrow F$:

$$\begin{aligned} [\pi_0^{(n)} id_F]_0(c_0) &= [id_F]_0(c_0) \\ &= id_{F c_0} \\ &= id_{[\pi_0^{(n)} F] c_0} \end{aligned}$$

and also

$$\begin{aligned} [\pi_0^{(n)} id_F]_1^{c_0, c'_0} &= \pi_0^{(n-1)}([id_F]_1^{c_0, c'_0}) \\ &= id_{\pi_0^{(n-1)}(F_1^{c_0, c'_0})} \\ &= id_{[\pi_0^{(n)} F]_1^{c_0, c'_0}}. \end{aligned}$$

2. We prove the statement for reduced left-composition. Suppose 2-morphism α as above, and morphism $N : \mathbb{B} \rightarrow \mathbb{C}$ be given.

When $n > 2$, for any objects b_0 of \mathbb{B} , one has

$$\begin{aligned} [\pi_0^{(n)}(N \bullet_L^0 \alpha)]_0(b_0) &\stackrel{(1)}{=} [N \bullet_L^0 \alpha]_0(b_0) \\ &\stackrel{(2)}{=} \alpha_0(N(b_0)) \\ &\stackrel{(3)}{=} [\pi_0^{(n)} \alpha]_0(N(b_0)) \\ &\stackrel{(4)}{=} [\pi_0^{(n)} \alpha]_0([\pi_0^{(n)} N](b_0)) \\ &\stackrel{(5)}{=} [\pi_0^{(n)} N \bullet_L^0 \pi_0^{(n)} \alpha]_0(b_0) \end{aligned}$$

where (1), (3) and (4) hold by definition of $\pi_0^{(n)}$, (2) and (5) by definition of reduced left-composition,

Moreover, let us choose two objects b_0 and b'_0 in \mathbb{B} . Then one has

$$\begin{aligned} [\pi_0^{(n)}(N \bullet_L^0 \alpha)]_1^{b_0, b'_0} &\stackrel{(1)}{=} \pi_0^{(n-1)}([N \bullet_L^0 \alpha]_1^{b_0, b'_0}) \\ &\stackrel{(2)}{=} \pi_0^{(n-1)}(N_1^{b_0, b'_0} \bullet_L^0 \alpha_1^{N b_0, N b'_0}) \\ &\stackrel{(3)}{=} \pi_0^{(n-1)}(N_1^{b_0, b'_0}) \bullet_L^0 \pi_0^{(n-1)}(\alpha_1^{N b_0, N b'_0}) \\ &\stackrel{(4)}{=} [\pi_0^{(n)} N]_1^{b_0, b'_0} \bullet_L^0 [\pi_0^{(n)} \alpha]_1^{\pi_0^{(n)} N(b_0), \pi_0^{(n)} N(b'_0)} \end{aligned}$$

where (1) and (4) hold by definition of $\pi_0^{(n)}$, (2) by definition of reduced left-composition and (3) by induction hypothesis.

When $n = 2$ the calculation can be carried on similarly, as we did for vertical composites above.

Finally, concerning reduced right-composition, the proof is similar, as the definition.

□

4.3.5 π_0 commutes with (finite) products

We will show that $\pi_0^{(n)}$ preserves binary products and the terminal object.

Proposition 4.19. *Let \mathbb{C} and \mathbb{D} be n -groupoids. Then*

1. $\pi_0^{(n)}(\mathbb{C} \times \mathbb{D}) = \pi_0^{(n)}\mathbb{C} \times \pi_0^{(n)}\mathbb{D}$
2. $\pi_0^{(n)}\left(\mathbb{I}_{\binom{n}{n}}\right) = \mathbb{I}_{\binom{n-1}{n-1}}$

Proof. 1. Consider the following equalities:

$$\begin{aligned}
 [\pi_0^{(n)}(\mathbb{C} \times \mathbb{D})]_0 &\stackrel{(1)}{=} \pi_0^{(n-1)}([\mathbb{C} \times \mathbb{D}]_0) \\
 &\stackrel{(2)}{=} \pi_0^{(n-1)}(\mathbb{C}_0 \times \mathbb{D}_0) \\
 &\stackrel{(3)}{=} \pi_0^{(n-1)}\mathbb{C}_0 \times \pi_0^{(n-1)}\mathbb{D}_0 \\
 &\stackrel{(4)}{=} [\pi_0^{(n)}\mathbb{C}]_0 \times [\pi_0^{(n)}\mathbb{D}]_0
 \end{aligned}$$

Moreover, let objects (c_0, d_0) and (c'_0, d'_0) of $\mathbb{C} \times \mathbb{D}$ be given. Then

$$\begin{aligned}
 [\pi_0^{(n)}(\mathbb{C} \times \mathbb{D})]_1^{(c_0, d_0), (c'_0, d'_0)} &\stackrel{(1)}{=} \pi_0^{(n-1)}\left([\mathbb{C} \times \mathbb{D}]_1^{(c_0, d_0), (c'_0, d'_0)}\right) \\
 &\stackrel{(2)}{=} \pi_0^{(n-1)}\left(\mathbb{C}_1^{c_0, c'_0} \times \mathbb{D}_1^{d_0, d'_0}\right) \\
 &\stackrel{(3)}{=} \pi_0^{(n-1)}(\mathbb{C}_1^{c_0, c'_0}) \times \pi_0^{(n-1)}(\mathbb{D}_1^{d_0, d'_0}) \\
 &\stackrel{(4)}{=} [\pi_0^{(n)}\mathbb{C}]_1^{c_0, c'_0} \times [\pi_0^{(n)}\mathbb{D}]_1^{d_0, d'_0}
 \end{aligned}$$

In both cases, (1) and (4) follow from the definition of $\pi_0^{(n)}$, (2) holds by definition of products and (3) by induction.

2. Consider the following equalities:

$$\begin{aligned}
 \left[\pi_0^{(n)}\left(\mathbb{I}_{\binom{n}{n}}\right)\right]_0 &\stackrel{(1)}{=} \left[\mathbb{I}_{\binom{n}{n}}\right]_0 \\
 &\stackrel{(2)}{=} \{*\} \\
 &\stackrel{(3)}{=} \left[\mathbb{I}_{\binom{n-1}{n-1}}\right]_0 \\
 &\stackrel{(4)}{=} \left[\pi_0^{(n)}\left(\mathbb{I}_{\binom{n}{n-1}}\right)\right]_0
 \end{aligned}$$

where (1) and (4) hold by definition of $\pi_0^{(n)}$, (2) and (3) hold by definition of terminal n-category.

$$\begin{aligned}
\left[\pi_0^{(n)} \left(\mathbb{I}_{(n)} \right) \right]_1^{*,*} &\stackrel{(1)}{=} \pi_0^{(n-1)} \left(\left[\mathbb{I}_{(n)} \right]_1^{*,*} \right) \\
&\stackrel{(2)}{=} \pi_0^{(n-1)} \left(\mathbb{I}_{(n-1)} \right) \\
&\stackrel{(3)}{=} \mathbb{I}_{(n-2)} \\
&\stackrel{(4)}{=} \left[\mathbb{I}_{(n-1)} \right]_1^{*,*}
\end{aligned}$$

where (1) and (4) follow from the definition of $\pi_0^{(n)}$, (2) holds by definition of terminal n-category and (3) by induction. \square

4.3.6 π_0 preserves equivalences

Proposition 4.20. *Let $F : \mathbb{C} \rightarrow \mathbb{D}$ be an equivalence of n -groupoids. Then $\pi_0^{(n)} F : \pi_0^{(n)} \mathbb{C} \rightarrow \pi_0^{(n)} \mathbb{D}$ is an equivalence of $(n-1)$ -groupoids.*

Proof. If $n = 1$, then F is an equivalence of categories, then $\pi_0' F$ is clearly an isomorphism. Hence we may well suppose $n > 1$.

1. Let objects c_0, c'_0 of \mathbb{C} be given. Then, by definition, $[\pi_0^{(n)} F]_1^{c_0, c'_0} = \pi_0^{(n-1)}(F_1^{c_0, c'_0})$. This is an equivalence of $(n-2)$ -groupoids, since $F_1^{c_0, c'_0}$ is an equivalence of $(n-1)$ -groupoids and $\pi_0^{(n-1)}$ preserves equivalences by induction.
2. Let an object d_0 of $\pi_0^{(n)} \mathbb{D}$ be given. This is indeed an object of \mathbb{D} , hence there exists a pair

$$(c_0, d_1 : Fc_0 \rightarrow d_0)$$

with d_1 being an equivalence in \mathbb{D} .

Now, c_0 is also an object of $\pi_0^{(n)} \mathbb{C}$, and d_1 (eventually $\{d_1\}_\sim$ if $n = 2$) is a 1-cell (hence an equivalence) of the $(n-1)$ groupoid $\pi_0^{(n)} \mathbb{D}$. \square

4.3.7 A remark on π_0

By the discussion in the previous section, $\pi_0^{(n)}$ is clearly extendible to a functor

$$[n\mathbf{Cat}] \rightarrow [(n-1)\mathbf{Cat}]$$

on underlying categories. Difficulties arise when we try to extend it further to a sesqui-functor. In fact, even for $n = 1$ our definition fails, since a

natural transformation $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$ does not imply that the maps $\pi_0 F \Rightarrow \pi_0 G : \pi_0 \mathbb{C} \rightarrow \pi_0 \mathbb{D}$ are equal. This is the case when α is a natural isomorphism, *i.e.* for every c_0 in \mathbb{C} , α_{c_0} is an isomorphism.

This case may be generalized in order to get a sesqui-functor on n -categories that remove the *obstruction* by considering only n -natural equivalences:

$$\pi_0^{(n)} : n\mathbf{Cat}_{eq} \rightarrow (n-1)\mathbf{Cat}_{eq}$$

This gives a chance to develop the theory in a more general case.

On the other hand, if one wants to keep into account *all* n -transformations, one direction can be to consider a generalization of *connected-components* functor, rather than *iso-classes* functor. At the moment, we have not explored this perspective since it seems to give rise to a completely different theory, not consistent with low-dimensional problems we aim to generalize.

4.4 The discretizer

Purpose of this section is to introduce the family of sesqui-functors $\{D^{(n)}\}_{n \in \mathbb{N}}$ that extends the *discrete-groupoid* functor $\mathbf{Set} \rightarrow \mathbf{Gpd}$.

Definition/Proposition 4.21. *For any integer $n > 0$, there exists a n -discrete sesqui-functor*

$$D^{(n)} : (n-1)\mathbf{Gpd} \rightarrow n\mathbf{Gpd}$$

according to the following recursive definition.

Moreover it commutes with finite products and it preserves equivalences

$$\boxed{n = 1}$$

$$D^{(1)} : \mathbf{Set} \rightarrow \mathbf{Gpd}$$

is the functor (= trivial sesqui-functor) that assigns to a set C the discrete groupoid $D(C)$, *i.e.* with objects the elements of C and only identity arrows.

$$\boxed{n > 1}$$

4.4.1 $D^{(n)}$ on objects and morphisms

Let a $(n-1)$ -groupoid \mathbb{C} be given. Then $D^{(n)}\mathbb{C} = ([D^{(n)}\mathbb{C}]_0, [D^{(n)}\mathbb{C}]_1(-, -))$, where

- $[D^{(n)}\mathbb{C}]_0 = C_0$
- for every pair c_0, c'_0 in $[D^{(n)}\mathbb{C}]_0$,

$$[D^{(n)}\mathbb{C}]_1(c_0, c'_0) = D^{(n-1)}(\mathbb{C}_1(c_0, c'_0))$$

For a triple of objects c_0, c'_0, c''_0 , composition is the dotted arrow:

$$\begin{array}{ccc}
 [D^{(n)}\mathbb{C}]_1(c_0, c'_0) \times [D^{(n)}\mathbb{C}]_1(c'_0, c''_0) & \xrightarrow{D^{(n)}\mathbb{C}_0} & [D^{(n)}\mathbb{C}]_1(c_0, c''_0) \\
 \parallel \scriptstyle{def} & & \parallel \scriptstyle{def} \\
 D^{(n-1)}(\mathbb{C}_1(c_0, c'_0)) \times D^{(n-1)}(\mathbb{C}_1(c'_0, c''_0)) & & \\
 \parallel \scriptstyle{(1)} & & \parallel \scriptstyle{def} \\
 D^{(n-1)}(\mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c''_0)) & \xrightarrow{D^{(n-1)}(\mathbb{C}_0)} & D_0^{(n-1)}(\mathbb{C}_1(c_0, c''_0))
 \end{array}$$

For an object c_0 , unit morphism is the dotted arrow:

$$\begin{array}{ccc}
 \mathbb{I}_{(n)} & \xrightarrow{\quad u(c_0) \quad} & [D_0^{(n)}\mathbb{C}]_1(c_0, c_0) \\
 (2) \parallel & & \parallel_{def} \\
 D_0^{(n-1)}(\mathbb{I}_{(n-1)}) & \xrightarrow{\quad D_0^{(n-1)}(u(c_0)) \quad} & D_0^{(n-1)}(\mathbb{C}_1(c_0, c_0))
 \end{array}$$

Equalities (1) and (2) hold because $D^{(n-1)}$ commutes with (finite) products by inductive hypothesis.

Let a $(n-1)$ functor $F : \mathbb{C} \rightarrow \mathbb{D}$ be given. Then, $D^{(n)}F = ([D^{(n)}F]_0, [D^{(n)}F]_1^-, -)$, where

- $[D^{(n)}F]_0 = F_0$
- for every pair c_0, c'_0 in $[D^{(n)}\mathbb{C}]_0$

$$[D^{(n)}F]_1^{c_0, c'_0} = D^{(n-1)}(F_1^{c_0, c'_0})$$

Notice that, since definitions of $\pi_0^{(n)}$ and of $D^{(n)}$ are formally identical, proving that all above is consistent is a matter of a syntactical substitution of the first with the second in the corresponding proofs concerning $\pi_0^{(n)}$. Hence we have defined the following functor between underlying categories:

$$[D^{(n)}] : [(\mathbf{n} - \mathbf{1})\mathbf{Gpd}] \rightarrow [\mathbf{nGpd}]$$

4.4.2 $D^{(n)}$ on 2-morphisms

Unlike that of $\pi_0^{(n)}$, the definition of $D^{(n)}$ is straightforward since the beginning of induction.

Let $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$ in \mathbf{nGpd} be given. As usual, $D^{(n)}(\alpha) = ([D^{(n)}\alpha]_0, [D^{(n)}\alpha]_1^-, -)$ where

- $[D^{(n)}\alpha]_0 = \alpha_0$
- for every pair of objects c_0, c'_0 in $[D\mathbb{C}]_0$

$$[D^{(n)}\alpha]_1^{c_0, c'_0} = D^{(n-1)}(\alpha_1^{c_0, c'_0})$$

Modulo the syntactical conversion mentioned above, we can prove that $D^{(n)}$ is a sesqui-functor that commutes with products and preserves equivalences. Hence it is well defined on n -groupoids.

4.4.3 A remark on $D^{(n)}$

Differently from $\pi_0^{(n)}$, the definition of $D^{(n)}$ extends with no changes to n -categories. In fact it *lives* more naturally in an n -categorical setting, and our definition is just its restriction to n -groupoids

4.5 The adjunction $\pi_0^{(n)} \dashv D^{(n)}$

4.5.1 In low dimension

The following adjunction has been extensively studied by category-theorist:

$$\mathbf{Gpd} \begin{array}{c} \xrightarrow{\pi_0} \\ \perp \\ \xleftarrow{D} \end{array} \mathbf{Set}$$

Let us describe it briefly, as it will be the first step of an inductive definition for general n -groupoids.

co-unit Given a set S , it is clear that $\pi_0(D(S)) = S$, hence co-unit ϵ is the identity.

unit For a groupoid \mathbb{C} , the unit

$$\eta = \eta_{\mathbb{C}} : \mathbb{C} \rightarrow D(\pi_0(\mathbb{C}))$$

is the *projection* given by $\eta_{\mathbb{C}}(c_0) = [c_0]_{\sim}$, and for $c_1 : c_0 \rightarrow c'_0$, $\eta_{\mathbb{C}}(c_1) = id_{[c_0]_{\sim}} = id_{[c'_0]_{\sim}}$

4.5.2 The general setting

Here and in the following, let an integer $n > 1$ be given.

For an $(n-1)$ -groupoid \mathbb{S} , the co-unit of the adjunction is still the identity. In fact $\pi_0^{(n)}(D^{(n)}(\mathbb{S})) = \mathbb{S}$.

Proof.

$$[\pi_0^{(n)}(D^{(n)}(\mathbb{S}))]_0 = [D^{(n)}(\mathbb{S})]_0 = \mathbb{S}_0$$

by definition, as $n > 1$. On the other side, for objects s_0, s'_0 one has

$$[\pi_0^{(n)}(D^{(n)}(\mathbb{S}))]_1(s_0, s'_0) = \pi_0^{(n-1)}([D^{(n)}(\mathbb{S})]_1(s_0, s'_0)) = \pi_0^{(n-1)}(D^{(n-1)}(\mathbb{S}_1(s_0, s'_0))) = \mathbb{S}_1(s_0, s'_0)$$

where the last equality is precisely the induction hypothesis.

Same argument holds for morphisms. □

Concerning the unit of the adjunction, we state the following

Definition 4.22. Let us fix n -groupoid \mathbb{C} . Then

$$\eta_{\mathbb{C}}^{(n)} : \mathbb{C} \rightarrow D^{(n)}(\pi_0^{(n)}(\mathbb{C}))$$

consists of the following data:

- $[\eta_{\mathbb{C}}^{(n)}]_0 = id_{\mathbb{C}_0} : \mathbb{C}_0 \rightarrow [D^{(n)}(\pi_0^{(n)}(\mathbb{C}))]_0 = \mathbb{C}_0$
- for any pair c_0, c'_0 of objects of \mathbb{C} , $[\eta_{\mathbb{C}}^{(n)}]_1^{c_0, c'_0}$ is the dotted arrow below:

$$\begin{array}{ccc} \mathbb{C}_1(c_0, c'_0) & \cdots \cdots \cdots \rightarrow & [D^{(n)}(\pi_0^{(n)}(\mathbb{C}))]_1(c_0, c'_0) \\ & \searrow \eta_{\mathbb{C}_1(c_0, c'_0)}^{(n-1)} & \parallel \\ & & D^{(n-1)}([\pi_0^{(n)}(\mathbb{C})]_1(c_0, c'_0)) \\ & & \parallel \\ & & D^{(n-1)}(\pi_0^{(n-1)}(\mathbb{C}_1(c_0, c'_0))) \end{array}$$

Claim 4.23. The pair $\langle id_{\mathbb{C}_0}, \eta_{\mathbb{C}_1(c_0, c'_0)}^{(n-1)} \rangle$ is a n -functor.

Proof. The diagrams marked (i) and (ii) express functoriality w.r.t. composition and units respectively, for any triple c_0, c'_0, c''_0 of objects of \mathbb{C} :

$$\begin{array}{ccc} \mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c''_0) & \xrightarrow{\mathbb{C}_o} & \mathbb{C}_1(c_0, c''_0) \\ \downarrow \eta_{\mathbb{C}_1(c_0, c'_0)}^{(n-1)} \times \eta_{\mathbb{C}_1(c'_0, c''_0)}^{(n-1)} & \text{(i)} & \downarrow \eta_{\mathbb{C}_1(c_0, c''_0)}^{(n-1)} \\ D^{(n-1)}(\pi_0^{(n-1)}(\mathbb{C}_1(c_0, c'_0))) \times D^{(n-1)}(\pi_0^{(n-1)}(\mathbb{C}_1(c'_0, c''_0))) & \searrow D^{(n)}\pi_0^{(n)}\mathbb{C}_o & \\ \parallel & & \downarrow \\ D^{(n-1)}(\pi_0^{(n-1)}(\mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c''_0))) & \xrightarrow{D^{(n-1)}\pi_0^{(n-1)}(\mathbb{C}_o)} & D^{(n-1)}(\pi_0^{(n-1)}(\mathbb{C}_1(c_0, c''_0))) \end{array}$$

$$\begin{array}{ccc} \mathbb{I}_{(n-1)} & \xrightarrow{\mathbb{C}_u(c_0)} & \mathbb{C}_1(c_0, c_0) \\ \parallel & \searrow D^{(n)}\pi_0^{(n)}\mathbb{C}_u(c_0) & \downarrow \eta_{\mathbb{C}_1(c_0, c_0)}^{(n-1)} \\ D^{(n-1)}(\pi_0^{(n-1)}(\mathbb{I}_{(n-1)})) & \xrightarrow{D^{(n-1)}\pi_0^{(n-1)}(\mathbb{C}_u(c_0))} & D^{(n-1)}(\pi_0^{(n-1)}(\mathbb{C}_1(c_0, c_0))) \end{array}$$

(ii)

Lower triangles commute by definition. Commutativity of external diagrams will be proved by finite induction in the following lemmas. This will imply that (i) and (ii) commute. \square

First we need the following conventional

Notation 4.24. Given the n -category \mathbb{C} , for $0 \leq s < m \leq n$ we write

$$c_m : c_s = \Rightarrow c'_s$$

meaning

$$c_m : c_{m-1} \rightarrow c'_{m-1} : \cdots : c_{s+1} \rightarrow c'_{s+1} : c_s \rightarrow c'_s$$

are m -cell, $(m-1)$ -cells, \dots , $(s+1)$ -cells, s -cells of \mathbb{C} .

Furthermore we inductively define

$$\mathbb{C}_m(c_{m-1}, c'_{m-1}) := [\mathbb{C}_{m-1}(c_{m-2}, c'_{m-2})]_1(c_{m-1}, c'_{m-1})$$

being $\mathbb{C}_1(c_0, c'_0)$ given by the definition of n -category.

Lemma 4.25. Given

$$c_{n-j-1}, k_{n-j-1} : c_0 = \Rightarrow c'_0, \quad c'_{n-j-1}, k'_{n-j-1} : c'_0 = \Rightarrow c''_0$$

the following diagram commutes in j -Gpd:

$$\begin{array}{ccc}
 \mathbb{C}_{n-j}(c_{n-j-1}, k_{n-j-1}) \times \mathbb{C}_{n-j}(c'_{n-j-1}, k'_{n-j-1}) & \xrightarrow{\circ^0} & \mathbb{C}_{n-j}(c_{n-j-1} \circ^0 c'_{n-j-1}, k_{n-j-1} \circ^0 k'_{n-j-1}) \\
 \eta_{\mathbb{C}_{n-j}(\diamond, \diamond)}^{(j)} \times \eta_{\mathbb{C}_{n-j}(\diamond, \diamond)}^{(j)} \downarrow & & \downarrow \eta_{\mathbb{C}_{n-j}(\diamond, \diamond)}^{(j)} \\
 D^{(j)}\pi_0^{(j)}(\mathbb{C}_{n-j}(c_{n-j-1}, k_{n-j-1})) \times D^{(j)}\pi_0^{(j)}(\mathbb{C}_{n-j}(c'_{n-j-1}, k'_{n-j-1})) & & D^{(j)}\pi_0^{(j)}(\mathbb{C}_{n-j}(c_{n-j-1} \circ^0 c'_{n-j-1}, k_{n-j-1} \circ^0 k'_{n-j-1})) \\
 \parallel & \nearrow D^{(j)}\pi_0^{(j)}(\circ^0) & \\
 D^{(j)}\pi_0^{(j)}(\mathbb{C}_{n-j}(c_{n-j-1}, k_{n-j-1}) \times \mathbb{C}_{n-j}(c'_{n-j-1}, k'_{n-j-1})) & &
 \end{array} \tag{4.1}$$

where we write (\diamond, \diamond) when substitutes are clear from the context.

Proof. By finite induction over j . Diagram above for $j = 1$ is the following square, of groupoids and functors

$$\begin{array}{ccc}
\mathbb{C}_{n-1}(c_{n-2}, k_{n-2}) \times \mathbb{C}_{n-1}(c'_{n-2}, k'_{n-2}) & \xrightarrow{\circ^0} & \mathbb{C}_{n-1}(c_{n-2} \circ^0 c'_{n-2}, k_{n-2} \circ^0 k'_{n-2}) \\
\downarrow \eta_{\mathbb{C}_{n-1}(\circ, \circ)}^{(1)} \times \eta_{\mathbb{C}_{n-1}(\circ, \circ)}^{(1)} & & \downarrow \eta_{\mathbb{C}_{n-1}(\circ, \circ)}^{(1)} \\
D^{(1)}\pi_0^{(1)}(\mathbb{C}_{n-1}(c_{n-2}, k_{n-2})) \times D^{(1)}\pi_0^{(1)}(\mathbb{C}_{n-1}(c'_{n-2}, k'_{n-2})) & & D^{(1)}\pi_0^{(1)}(\mathbb{C}_{n-1}(c_{n-2} \circ^0 c'_{n-2}, k_{n-2} \circ^0 k'_{n-2})) \\
\parallel & & \downarrow \\
D^{(1)}\pi_0^{(1)}(\mathbb{C}_{n-1}(c_{n-2}, k_{n-2}) \times \mathbb{C}_{n-1}(c'_{n-2}, k'_{n-2})) & \xrightarrow{D^{(1)}\pi_0^{(1)}(\circ^0)} & D^{(1)}\pi_0^{(1)}(\mathbb{C}_{n-1}(c_{n-2} \circ^0 c'_{n-2}, k_{n-2} \circ^0 k'_{n-2}))
\end{array}$$

For this to commute, it must commute on objects and on arrows. In order to prove this, let us consider

$$c_n : c_{n-1} \rightarrow k_{n-1} : c_{n-2} \rightarrow k_{n-2}$$

and

$$c'_n : c'_{n-1} \rightarrow k'_{n-1} : c'_{n-2} \rightarrow k'_{n-2}$$

Moreover, let us recall that

$$\eta_{\mathbb{X}}^{(1)} := \langle [Id_{\mathbb{X}_0}]_{\sim}, Id_{[dom(-)]_{\sim}} \rangle$$

and apply: for the pair (c_{n-1}, c'_{n-1}) , on the lower-left one has

$$\begin{aligned}
(c_{n-1}, c'_{n-1}) &\xrightarrow{[\eta^{(1)}]_0 \times [\eta^{(1)}]_0} ([c_{n-1}]_{\sim}, [c'_{n-1}]_{\sim}) \\
&= [(c_{n-1}, c'_{n-1})]_{\sim} \xrightarrow{[\circ^0]_{\sim}} [c_{n-1} \circ^0 c'_{n-1}]_{\sim},
\end{aligned}$$

where on the upper-right

$$\begin{aligned}
(c_{n-1}, c'_{n-1}) &\xrightarrow{\circ^0} c_{n-1} \circ^0 c'_{n-1} \\
&= c_{n-1} \circ^0 c'_{n-1} \xrightarrow{[\eta^{(1)}]_0} [c_{n-1} \circ^0 c'_{n-1}]_{\sim}
\end{aligned}$$

Similarly for arrows, on the lower-left one has

$$\begin{aligned}
(c_n, c'_n) &\xrightarrow{[\eta^{(1)}]_1 \times [\eta^{(1)}]_1} (Id_{[c_{n-1}]_{\sim}}, Id_{[c'_{n-1}]_{\sim}}) \\
&= Id_{[(c_{n-1}, c'_{n-1})]_{\sim}} \xrightarrow{[\circ^0]_{\sim}} Id_{[c_{n-1} \circ^0 c'_{n-1}]_{\sim}},
\end{aligned}$$

while on the upper-right

$$\begin{aligned} (c_n, c'_n) &\xrightarrow{\circ^0} c_n \circ^0 c'_n \\ &= c_n \circ^0 c'_n \xrightarrow{[\eta^{(1)}]_0} Id_{[c_{n-1} \circ^0 c'_{n-1}] \sim} , \end{aligned}$$

and this completes the case $j = 1$.

Now let us assume, as induction hypothesis, that Lemma holds for $j - 1$. We will prove it holds for j .

To this end, suppose we are given

$$c_{n-j+1} : c_{n-j} \rightarrow k_{n-j} : c_{n-j-1} \rightarrow k_{n-j-1}$$

and

$$c'_{n-j+1} : c'_{n-j} \rightarrow k'_{n-j} : c'_{n-j-1} \rightarrow k'_{n-j-1}.$$

For the pair (c_{n-j}, c'_{n-j}) , on the lower-left one has

$$\begin{aligned} (c_{n-j}, c'_{n-j}) &\xrightarrow{[\eta^{(j)}]_0 \times [\eta^{(j)}]_0} (c_{n-j}, c'_{n-j}) \\ &= (c_{n-j}, c'_{n-j}) \xrightarrow{\circ^0} c_{n-j} \circ^0 c'_{n-j} , \end{aligned}$$

where on the upper-right

$$\begin{aligned} (c_{n-j}, c'_{n-j}) &\xrightarrow{\circ^0} c_{n-j} \circ^0 c'_{n-j} \\ &= c_{n-j} \circ^0 c'_{n-j} \xrightarrow{[\eta^{(j)}]_0} c_{n-j} \circ^0 c'_{n-j} \end{aligned}$$

For homs, let us say that

$$\left[\text{Diagram (4.1) for } j \right]_1 \left((c_{n-j}, c'_{n-j}), (k_{n-j}, k'_{n-j}) \right)$$

is indeed $\left[\text{Diagram (4.1) for } (j - 1) \right]$. Induction completes the proof. \square

In the same way one can prove the following

Lemma 4.26. *Given*

$$c_{n-j-1} : c_0 = \Rightarrow c_0 ,$$

the following diagram commutes in $j\text{-}\mathbf{Gpd}$:

$$\begin{array}{ccc} \mathbb{I}_{(j)} & \xrightarrow{c_{u^0(c_0)}} & \mathbb{C}_{n-j}(c_{n-j-1}, c_{n-j-1}) \\ \parallel & & \downarrow \eta_{\mathbb{C}_{n-j}(c_{n-j-1}, c_{n-j-1})}^j \\ D^{(j)}\pi_0^{(j)}\left(\mathbb{I}_{(j)}\right) & \xrightarrow{D^{(j)}\pi_0^{(j)}(c_{u^0(c_0)})} & D^{(j)}\pi_0^{(j)}(\mathbb{C}_{n-j}(c_{n-j-1}, c_{n-j-1})) \end{array} \quad (4.2)$$

Finally we can conclude the

Proof. (of *Claim 4.23*). The first of the diagrams of *Claim 4.23* amounts to diagram (4.1) with $j = (n - 1)$, the second amounts to diagram (4.2) again with $j = (n - 1)$. \square

Proposition 4.27.

$$\eta^{(n)} : id_{\mathbf{Gpd}} \Rightarrow D^{(n)}(\pi_0^{(n)}(-))$$

is a natural transformation of sesqui-functors.

Proof. It suffices to show that, for any $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$,

$$\alpha \bullet^0 \eta_{\mathbb{D}}^{(n)} = \eta_{\mathbb{C}}^{(n)} \bullet^0 D^{(n)}(\pi_0^{(n)}(\alpha))$$

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\eta_{\mathbb{C}}} & D(\pi_0(\mathbb{C})) \\ \downarrow F & \searrow \alpha \Rightarrow & \downarrow D(\pi_0(F)) \\ & G & D(\pi_0(G)) \\ \downarrow \eta_{\mathbb{D}} & \swarrow D(\pi_0(\alpha)) \Rightarrow & \downarrow D(\pi_0(G)) \\ \mathbb{D} & \xrightarrow{\eta_{\mathbb{D}}} & D(\pi_0(\mathbb{D})) \end{array}$$

By induction on n .

$$\boxed{n = 1}$$

The adjunction of underlying categories and functors is well known. It extends plainly to sesqui-categories: in fact, in \mathbf{Gpd} , $\alpha(c_0)$'s are isomorphisms. Hence, $D'(\pi_0'(\alpha))$ is an equality of functors.

$$\boxed{n > 1}$$

On objects, let us fix c_0 in \mathbb{C}_0 . Then

$$[\alpha \bullet^0 \eta_{\mathbb{D}}^{(n)}]_0(c_0) = \eta_{\mathbb{D}}^{(n)}(\alpha(c_0)) = [\eta_{\mathbb{D}_1(Fc_0, Gc_0)}^{(n)}]_0(\alpha(c_0)) = \alpha(c_0)$$

(or $[\alpha(c_0)]_{\sim}$ if $n = 2$). On the other side,

$$[\eta_{\mathbb{C}}^{(n)} \bullet^0 D^{(n)}(\pi_0^{(n)} \alpha)]_0(c_0) = [D^{(n)}(\pi_0^{(n)} \alpha)]_0([\eta_{\mathbb{C}}^{(n)}]_0(c_0)) = [D^{(n)}(\pi_0^{(n)} \alpha)]_0(c_0) = [\pi_0^{(n)} \alpha]_0(c_0) = \alpha(c_0)$$

(or $[\alpha(c_0)]_{\sim}$ if $n = 2$).

On homs, let us fix one more object c'_0 . Then

$$[\alpha \bullet^0 \eta_{\mathbb{D}}^{(n)}]_1^{c_0, c'_0} = \alpha_1^{c_0, c'_0} \bullet^0 [\eta_{\mathbb{D}}^{(n)}]_1^{Fc_0, Gc'_0} = \alpha_1^{c_0, c'_0} \bullet^0 \eta_{\mathbb{D}_1(Fc_0, Gc'_0)}^{(n-1)}$$

on the other side,

$$[\eta_{\mathbb{C}}^{(n)} \bullet^0 D^{(n)}(\pi_0^{(n)} \alpha)]_1^{c_0, c'_0} = [\eta_{\mathbb{C}}^{(n)}]_1^{c_0, c'_0} \bullet^0 [D^{(n)}(\pi_0^{(n)} \alpha)]_1^{c_0, c'_0} = \eta_{\mathbb{C}_1(c_0, c'_0)}^{(n-1)} \bullet^0 D^{(n-1)}(\pi_0^{(n-1)}(\alpha_1^{c_0, c'_0}))$$

Equality of the two sides

$$\alpha_1^{c_0, c'_0} \bullet^0 \eta_{\mathbb{D}_1(Fc_0, Gc'_0)}^{(n-1)} = \eta_{\mathbb{C}_1(c_0, c'_0)}^{(n-1)} \bullet^0 D^{(n-1)}(\pi_0^{(n-1)}(\alpha_1^{c_0, c'_0}))$$

is given by induction hypothesis. \square

Theorem 4.28. *For every positive integer n ,*

$$\pi_0^{(n)} \dashv D^{(n)}$$

Proof. After the discussion above, triangular identities will be proved in the following form:

$$\begin{array}{ccc} D^{(n)} & \xRightarrow{\eta^{(n)} D^{(n)}} & D^{(n)} \pi_0^{(n)} D^{(n)} \\ & \searrow (i) & \parallel D^{(n)} \epsilon^{(n)} = D^{(n)} \\ & & D^{(n)} \end{array} \qquad \begin{array}{ccc} \pi_0^{(n)} & \xRightarrow{\pi_0^{(n)} \eta^{(n)}} & \pi_0^{(n)} D^{(n)} \pi_0^{(n)} \\ & \searrow (ii) & \parallel \epsilon^{(n)} \pi_0^{(n)} = \pi_0^{(n)} \\ & & \pi_0^{(n)} \end{array}$$

Diagram (i) commutes. In fact, for a $(n-1)$ -groupoid \mathbb{S} one has $\eta_{D^{(n)}\mathbb{S}}^{(n)} = id_{D^{(n)}\mathbb{S}}$.

By induction on n . For $n = 1$, $\mathbb{S} = S$ is a set and $\eta'_{D'S}$ is given by

$$[\eta'_{D'S}]_0 : s_0 \mapsto [s_0]_{\sim} = s_0$$

Since $D'S$ is a discrete category, it only has identity arrows, hence $[\eta'_{D'S}]_0 = id$ by functoriality.

For $n > 1$, by definition one has

$$[\eta_{D^{(n)}\mathbb{S}}]_0 = id_{[D^{(n)}\mathbb{S}]_0}$$

Moreover, for any pair of objects s_0, s'_0 of \mathbb{S} ,

$$\begin{aligned} [\eta_{D^{(n)}\mathbb{S}}]_1^{s_0, s'_0} &= \eta_{[D^{(n)}\mathbb{S}]_1}^{(n-1), s_0, s'_0} \\ &= \eta_{D^{(n-1)}(\mathbb{S}_1(s_0, s'_0))}^{(n-1)} \\ &\stackrel{(\clubsuit)}{=} id_{D^{(n-1)}(\mathbb{S}_1(s_0, s'_0))} \\ &= id_{[D^{(n)}\mathbb{S}]_1(s_0, s'_0)} \\ &= [id_{D^{(n)}\mathbb{S}}]_1(s_0, s'_0) \end{aligned}$$

where all equalities hold by definition, but (\clubsuit) that is given by induction hypothesis.

Diagram (ii) commutes. In fact, for a n -groupoid \mathbb{C} one has $\pi_0^{(n)}(\eta_{\mathbb{C}}^{(n)}) = id_{\pi_0^{(n)}\mathbb{C}}$.

By induction on n . For $n = 1$, *i.e.* for a groupoid \mathbb{C} , $\pi'_0(\eta'_{\mathbb{C}}) = [\pi'_0(\eta'_{\mathbb{C}})]_0$, and it is given by

$$[\pi'_0(\eta'_{\mathbb{C}})]_0 : [c_0]_{\sim} \mapsto [\pi'_0]_0([\eta'_{\mathbb{C}}]_0([c_0]_{\sim})) = \pi'_0([c_0]_{\sim}) = [c_0]_{\sim}$$

For $n > 1$, by definition one has

$$[\pi_0^{(n)}(\eta_{\mathbb{C}}^{(n)})]_0 = id_{[\pi_0^{(n)}\mathbb{C}]_0}$$

Moreover, for any pair of objects c_0, c'_0 of \mathbb{C} ,

$$\begin{aligned} [\pi_0^{(n)}(\eta_{\mathbb{C}}^{(n)})]_1^{c_0, c'_0} &= \pi_0^{(n-1)}([\eta_{\mathbb{C}}^{(n)}]_1^{c_0, c'_0}) \\ &= \pi_0^{(n-1)}(\eta_{\mathbb{C}_1(c_0, c'_0)}^{(n)}) \\ &\stackrel{(\clubsuit)}{=} id_{\pi_0^{(n-1)}(\mathbb{C}_1(c_0, c'_0))} \\ &= id_{[\pi_0^{(n)}\mathbb{C}]_1(c_0, c'_0)} \end{aligned}$$

where all equalities hold by definition, but (\clubsuit) that is given by induction hypothesis. □

4.6 n -Discrete h -pullbacks

An application of the adjunction $\pi_0^{(n)} \dashv D^{(n)}$ is the following useful result, just a special case of more general h -limits preservation property:

Lemma 4.29. *Sesqui-functor $D^{(n)} : (n-1)\mathbf{Gpd} \rightarrow n\mathbf{Gpd}$ preserves h -pullbacks.*

Proof. We omit the superscripts being always (n) . Let us consider D of the h -pullback (\mathbb{P}, P, Q, ϕ) in $(n-1)\mathbf{Gpd}$

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{Q} & \mathbb{C} \\ P \downarrow & \nearrow \phi & \downarrow G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array} \quad \xrightarrow{D} \quad \begin{array}{ccc} D\mathbb{P} & \xrightarrow{DQ} & D\mathbb{C} \\ DP \downarrow & \nearrow D\phi & \downarrow DG \\ D\mathbb{A} & \xrightarrow{DF} & D\mathbb{B} \end{array}$$

For the four-tuple $(\mathbb{Q}, M, N, \omega)$ over the base $\langle DF, DG \rangle$, we can now apply π_0

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{N} & D\mathbb{C} \\ M \downarrow & \nearrow \omega & \downarrow DG \\ D\mathbb{A} & \xrightarrow{DF} & D\mathbb{B} \end{array} \quad \xrightarrow{\pi_0} \quad \begin{array}{ccc} \pi_0\mathbb{Q} & \xrightarrow{\pi_0 N} & \mathbb{C} \\ \pi_0 M \downarrow & \nearrow \pi_0\omega & \downarrow G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array}$$

universal property for \mathbb{P} gives then a unique $L : \pi_0\mathbb{Q} \rightarrow \mathbb{P}$ such that

$$(i) \ L \bullet^0 P = \pi_0 M \quad (ii) \ L \bullet^0 Q = \pi_0 N \quad (iii) \ L \bullet^0 \phi = \pi_0 \omega$$

Hence the composition $\eta_{\mathbb{Q}} \bullet^0 DL$ witnesses the fact that $D(\mathbb{P})$ is an h -pullback. In fact

$$\begin{aligned} (i)' \quad & \eta_{\mathbb{Q}} \bullet^0 DL \bullet^0 DP = \eta_{\mathbb{Q}} \bullet^0 D(L \bullet^0 P) \stackrel{(i)}{=} \eta_{\mathbb{Q}} \bullet^0 \pi_0 M \stackrel{(a)}{=} M \bullet^0 \eta_{D\mathbb{A}} \stackrel{(b)}{=} M \\ (ii)' \quad & \eta_{\mathbb{Q}} \bullet^0 DL \bullet^0 DQ = \eta_{\mathbb{Q}} \bullet^0 D(L \bullet^0 Q) \stackrel{(ii)}{=} \eta_{\mathbb{Q}} \bullet^0 \pi_0 N \stackrel{(a)}{=} N \bullet^0 \eta_{D\mathbb{C}} \stackrel{(b)}{=} N \\ (iii)' \quad & \eta_{\mathbb{Q}} \bullet^0 DL \bullet^0 D\phi = \eta_{\mathbb{Q}} \bullet^0 D(L \bullet^0 \phi) \stackrel{(iii)}{=} \eta_{\mathbb{Q}} \bullet^0 \pi_0 \omega \stackrel{(a)}{=} \omega \bullet^0 \eta_{D\mathbb{C}} \stackrel{(b)}{=} \omega \end{aligned}$$

where (a) by naturality of η and (b) by triangular identities. \square

Hence we can say that the h -pullback of a n -discrete diagram is itself n -discrete.

4.7 Exact sequences of n -groupoids

4.7.1 Pointedness and h -fibers

A pointed n -category is simply a n -category \mathbb{C} with a chosen object $*_{\mathbb{C}}$.

A morphism of pointed n -categories $F : \mathbb{C} \rightarrow \mathbb{D}$ is a n -functor such that

$$F(*_{\mathbb{C}}) = *_{\mathbb{D}}$$

A 2-morphism of pointed n -functors $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$ is a natural n -transformation such that $\alpha_0(*_{\mathbb{C}}) = id_{*_{\mathbb{D}}} : *_{\mathbb{D}} \rightarrow *_{\mathbb{D}}$.

The data described above form a sesqui-category, sub-sesqui-category of $n\mathbf{Cat}$ that we will denote $n\mathbf{Cat}_*$. Similarly one defines the sesqui-category $n\mathbf{Gpd}_*$ of pointed n -groupoids, sub-sesqui-category of $n\mathbf{Gpd}$.

Subscripts of the *star* will be often omitted.

Notice that definitions of pointed morphism and of pointed 2-morphisms imply that $n\mathbf{Cat}_*$ and $n\mathbf{Gpd}_*$ are closed under finite products and h -pullbacks.

Definition 4.30. Given a morphism of n -groupoids $F : \mathbb{C} \rightarrow \mathbb{D}$, and an object d of \mathbb{D} , the past h -fiber $\mathbb{F}^{(p)}$ and the future h -fiber $\mathbb{F}^{(f)}$ of F over d are given by the following h -pullbacks resp.

$$\begin{array}{ccc} \mathbb{F}_{F,d}^{(p)} & \xrightarrow{!} & \mathbb{I} \\ E \downarrow & \swarrow \varepsilon & \downarrow [d] \\ \mathbb{C} & \xrightarrow{F} & \mathbb{D} \end{array} \quad \begin{array}{ccc} \mathbb{F}_{F,d}^{(f)} & \xrightarrow{E} & \mathbb{C} \\ ! \downarrow & \swarrow \varepsilon & \downarrow F \\ \mathbb{I} & \xrightarrow{[d]} & \mathbb{D} \end{array}$$

where $[d]$ is the constant functor over an object d , as usual.

Remark 4.31. Distinction between future and past h -fibers, makes more sense in n -categorical context. There, in fact, the h -fiber may come in different tastes: the lax-version that uses lax-2-morphisms, the pseudo-version, that uses equivalence transformations. Nevertheless, even for n -groupoids, keeping track of the direction of 2-morphisms is necessary in order to recognize the problem of coherently choosing inverses of cells, when this problem arises. Still, we will often omit superscripts (and subscripts), as they will be clear from diagrams.

Remark 4.32. The notion of strict fiber, or simply fiber, is recovered by the strict pullback (see Section 3.6.5).

4.7.2 Equivalences and h -surjective morphisms of n -groupoids

The notions of h -surjective morphism and of *equivalence* get simpler in the context of n -groupoid. Concerning both cases, it is the notion of essential surjectivity itself to be involved. In fact, for a n -groupoids morphism $L : \mathbb{A} \rightarrow \mathbb{K}$, to be essentially surjective on objects amounts to the following property:

Essential surjectivity (1) for any object k_0 of \mathbb{K} , there exists a pair

$$(a_0, k_1 : La_0 \rightarrow k_0)$$

with a_0 object of \mathbb{A} and k_1 1-cell of \mathbb{K} .

That is: it is no longer necessary to ask for k_1 to be an equivalence, since every cell in a n -groupoid is indeed an equivalence.

With the notion of h -fiber in mind, we can further reformulate the notion of essential surjectivity.

Essential surjectivity (2) for any object k_0 of \mathbb{K} , the h -fiber

$$\mathbb{F}_{L, k_0}^{(p)}$$

is not empty.

Finally, in pointed case, fibers assume a special meaning, as referred by the following

Definition 4.33. Let $G : \mathbb{B} \rightarrow \mathbb{C}$ be a morphism of n -groupoids. Then the fiber $\mathbb{K} = \mathbb{F}_{G,*}^{(p)}$

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \Downarrow \kappa & \swarrow & \\ \mathbb{K} & \xrightarrow{K} & \mathbb{B} & \xrightarrow{G} & \mathbb{C} \end{array}$$

is called (past) the h -kernel of G .

We will call universal property of h -kernels, the universal property of pullbacks specialized for these kind of pullbacks.

4.7.3 Exact sequences

Definition 4.34. Let the following diagram in \mathbf{nGpd}_* be given:

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \Downarrow \varepsilon & \swarrow & \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} & \xrightarrow{G} & \mathbb{C} \end{array}$$

We call the triple (F, ε, G) exact in \mathbb{B} if the comparison n -functor $L : \mathbb{A} \rightarrow \mathbb{K}$, given by the universal property of the h -kernel (\mathbb{K}, K, κ) , is h -surjective.

$$\begin{array}{ccccc} \mathbb{A} & & & 0 & \\ \downarrow L & \searrow F & & \Downarrow \varepsilon & \swarrow \\ & \mathbb{B} & \xrightarrow{G} & \mathbb{C} & \\ & \nearrow K & & \Downarrow \kappa & \\ \mathbb{K} & & & 0 & \end{array}$$

This notion of exactness is a straightforward extensions of the notion introduced by Vitale in [Vit02].

Of course it reduces to usual exactness for pointed sets and group. Moreover it is preserved by one-point suspension and by discretization, hence an exact sequence of groups may be considered as an exact sequence of one-point groupoids, as well as an exact sequence of pointed discrete groupoids (with a group structure).

In the categorical group (pointed groupoid) situation, it has shown its usefulness in extending homological algebraic structures in a 1-dimensional context.

4.7.4 π_0 preserves exactness

In the following paragraphs we will show that, given a three-term exact sequence in $n\mathbf{Gpd}$, the sesqui-functor $\pi_0^{(n)}$ produces a three-term exact sequence in $(n-1)\mathbf{Gpd}$. Preliminary Lemmas clarify the relations between preservation of exactness and its main ingredients: h -surjectivity and the notion of h -pullback.

Lemma 4.35. *Let us consider the following h -pullback diagram:*

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{S} & \mathbb{Z} \\ R \downarrow & \nearrow \varepsilon & \downarrow H \\ \mathbb{B} & \xrightarrow{G} & \mathbb{C} \end{array}$$

The comparison $L : \pi_0^{(n)}\mathbb{P} \rightarrow \mathbb{Q}$ with the h -pullback of $\pi_0^{(n)}(G)$ and $\pi_0^{(n)}(H)$ is h -surjective.

$$\begin{array}{ccccc} \pi_0^{(n)}\mathbb{P} & & \xrightarrow{\pi_0^{(n)}S} & & \pi_0^{(n)}\mathbb{Z} \\ & \searrow \pi_0^{(n)}\varepsilon & \nearrow L & & \downarrow \pi_0^{(n)}H \\ & & \mathbb{Q} & \xrightarrow{Q} & \pi_0^{(n)}\mathbb{Z} \\ & \searrow \pi_0^{(n)}R & \nearrow P & & \downarrow \pi_0^{(n)}H \\ & & \pi_0^{(n)}\mathbb{B} & \xrightarrow{\pi_0^{(n)}G} & \pi_0^{(n)}\mathbb{C} \end{array}$$

Proof. By induction on n .

$$\boxed{n = 1}$$

The h -pullback \mathbb{P} has objects

$$(b_0, Gb_0 \xrightarrow{c_1} Hz_0, z_0)$$

and arrows

$$(b_1, =, z_1) : (b_0, c_1, z_0) \rightarrow (b'_0, c'_1, z'_0)$$

where the “=” stays for the commutative square

$$\begin{array}{ccc} Gb_0 & \xrightarrow{c_1} & Hz_0 \\ Gb_1 \downarrow & & \downarrow Hz_1 \\ Gb'_0 & \xrightarrow{c'_1} & Hz'_0 \end{array}$$

Hence the set $\pi'_0(\mathbb{P})$ has elements the classes

$$[b_0, c_1, z_0]_{\sim}.$$

On the other side, the set \mathbb{Q} is a usual pullback in **Set**. It has elements the pairs $([b_0]_{\sim}, [z_0]_{\sim})$ such that $\pi'_0 G([b_0]_{\sim}) = \pi'_0 H([z_0]_{\sim})$, *i.e.* $[Gb_0]_{\sim} = [Hz_0]_{\sim}$, *i.e.* such that there exists $c_1 : Gb_0 \rightarrow Hz_0$. Then the comparison

$$L = L_0 : [b_0, c_1, z_0]_{\sim} \mapsto ([b_0]_{\sim}, [z_0]_{\sim})$$

is clearly surjective.

$$\boxed{n = 2}$$

Now the h -pullback \mathbb{P} is a 2-groupoid with objects

$$(b_0, Gb_0 \xrightarrow{c_1} Hz_0, z_0).$$

Arrows are of the form

$$(b_1, c_2, z_1) : (b_0, c_1, z_0) \rightarrow (b'_0, c'_1, z'_0)$$

i.e.

$$\left(\begin{array}{ccccc} b_0 & Gb_0 & \xrightarrow{c_1} & Hz_0 & z_0 \\ b_1 \downarrow, & Gb_1 \downarrow & \nearrow c_2 & \downarrow Hz_1, & z_1 \downarrow \\ b'_0 & Gb'_0 & \xrightarrow{c'_1} & Hz'_0 & z'_0 \end{array} \right)$$

Finally 2-cells are of the form

$$(b_2, =, z_2) : (b_1, c_2, z_1) \Rightarrow (b'_1, c'_2, z'_1)$$

i.e.

$$\left(\begin{array}{ccccc} b_1 & Gb_1 \circ c'_1 & \xRightarrow{c_2} & c_1 \circ Hz_1 & z_1 \\ b_2 \Downarrow, & Gb_2 \circ c'_1 \Downarrow & & \Downarrow c_1 \circ Hz_2, & \Downarrow z_2 \\ b'_1 & Gb'_1 \circ c'_1 & \xRightarrow{c'_2} & c_1 \circ Hz'_1 & z'_1 \end{array} \right)$$

Therefore the groupoid $\pi_0''\mathbb{P}$ has objects (b_0, c_1, z_0) and arrows $[b_1, c_2, z_1]_\sim$.

On the other side, the groupoid \mathbb{Q} has objects

$$(b_0, Gb_0 \xrightarrow{[c_1]_\sim} Hz_0, z_0)$$

and arrows

$$([b_1]_\sim, =, [z_1]_\sim)$$

with $[b_1]_\sim : b_0 \rightarrow b'_0$ in $\pi_0''\mathbb{B}$ and $[z_1]_\sim : z_0 \rightarrow z'_0$ in $\pi_0''\mathbb{Z}$ such that

$$\begin{array}{ccc} Gb_0 & \xrightarrow{[c_1]_\sim} & Hz_0 \\ [Gb_1]_\sim \downarrow & (\heartsuit) & \downarrow [Hz_1]_\sim \\ Gb'_0 & \xrightarrow{[c'_1]_\sim} & Hz'_0 \end{array}$$

Hence the comparison

$$\begin{aligned} L & : (b_0, c_1, z_0) \mapsto (b_0, c_1, z_0) \\ & [b_1, c_2, z_1]_\sim \mapsto ([b_1]_\sim, [z_1]_\sim) \end{aligned}$$

is h -surjective. In fact it is an identity (hence strictly surjective) on objects, and full on homs. Let us fix a pair of objects (b_0, c_1, z_0) and (b'_0, c'_1, z'_0) in the domain, and an arrow $([b_1]_\sim, =, [z_1]_\sim)$ in \mathbb{Q} , where the “=” is the diagram (\heartsuit) above. Then $[c_1 \circ Hz_1]_\sim = [Gb_1 \circ c'_1]_\sim$ if, and only if, there exists

$$c_2 : c_1 \circ Hz_1 \rightarrow Gb_1 \circ c'_1.$$

In other words we get an arrow $[b_1, c_2, z_1]_\sim$ of $\pi_0''\mathbb{P}$ that L sends in $([b_1]_\sim, =, [z_1]_\sim)$, *i.e.* L is full.

$n > 2$

On objects, $L_0 : [\pi_0^{(n)}\mathbb{P}]_0 \rightarrow \mathbb{Q}_0$ is the identity.

In fact, for big n $[\pi_0^{(n)}\mathbb{P}]_0 = \mathbb{P}_0$, the last being the set-theoretical limit over the diagram

$$\begin{array}{ccccc} \mathbb{B}_0 & & \mathbb{C}_1 & & \mathbb{Z}_0 \\ & \searrow G_0 & \swarrow d \quad \searrow c & \swarrow H_0 & \\ & \mathbb{C}_0 & & \mathbb{C}_0 & \end{array}$$

Now, for $n > 2$ this diagram coincides with the one defining \mathbb{Q}_0 :

$$\begin{array}{ccccc} [\pi_0^{(n)}\mathbb{B}]_0 & & [\pi_0^{(n)}\mathbb{C}]_1 & & [\pi_0^{(n)}\mathbb{Z}]_0 \\ & \searrow [\pi_0^{(n)}G]_0 & \swarrow d \quad \searrow c & \swarrow [\pi_0^{(n)}H]_0 & \\ & [\pi_0^{(n)}\mathbb{C}]_0 & & [\pi_0^{(n)}\mathbb{C}]_0 & \end{array}$$

Universality of limits gives $L_0 = id$.

On homs, let us fix two objects $p_1 = (b_0, c_1, z_0)$ and $p'_1 = (b'_0, c'_1, z'_0)$ of $[\pi_0^{(n)} \mathbb{P}]_0 = \mathbb{P}_0$ and compute $L_1^{p_1, p'_1}$ as usual: by means of universal property of h -pullbacks. In fact the diagram

$$\begin{array}{ccccc}
 [\pi_0^{(n)} \mathbb{P}]_1(p_0, p'_0) & \xrightarrow{\quad [\pi_0^{(n)} S]_1 \quad} & & & \\
 \swarrow \scriptstyle L_1^{p_0, p'_0} & & & & \\
 \swarrow \scriptstyle [\pi_0^{(n)} \varepsilon]_1 & \searrow & \mathbb{Q}_1(p_0, p'_0) & \xrightarrow{\quad Q_1 \quad} & [\pi_0^{(n)} \mathbb{Z}]_1(z_0, z'_0) \\
 \downarrow \scriptstyle P_1 & & \downarrow & & \downarrow \scriptstyle [\pi_0^{(n)} H]_1 \\
 [\pi_0^{(n)} \mathbb{B}]_1(b_0, b'_0) & \xrightarrow{\quad [\pi_0^{(n)} G]_1 \quad} & [\pi_0^{(n)} \mathbb{C}]_1(Gb_0, Gb'_0) & \xrightarrow{\quad - \circ c'_1 \quad} & [\pi_0^{(n)} \mathbb{C}]_1(Gb_0, Hz'_0) \\
 \uparrow \scriptstyle [\pi_0^{(n)} R]_1 & & \uparrow \scriptstyle \sigma & & \downarrow \scriptstyle c_1 \circ - \\
 & & & & [\pi_0^{(n)} \mathbb{C}]_1(Hz_0, Hz'_0)
 \end{array}$$

is the same as (and determined by)

$$\begin{array}{ccccc}
 \pi_0^{(n-1)}(\mathbb{P}_1(p_0, p'_0)) & \xrightarrow{\quad \pi_0^{(n-1)} S_1 \quad} & & & \\
 \swarrow \scriptstyle L_1^{p_0, p'_0} & & & & \\
 \swarrow \scriptstyle \pi_0^{(n-1)}(\varepsilon_1) & \searrow & \mathbb{Q}_1(p_0, p'_0) & \xrightarrow{\quad Q_1 \quad} & [\pi_0^{(n)} \mathbb{Z}]_1(z_0, z'_0) \\
 \downarrow \scriptstyle P_1 & & \downarrow & & \downarrow \scriptstyle \pi_0^{(n-1)} H_1 \\
 \pi_0^{(n-1)}(\mathbb{B}_1(b_0, b'_0)) & \xrightarrow{\quad \pi_0^{(n-1)} G_1 \quad} & \pi_0^{(n-1)}(\mathbb{C}_1(Gb_0, Gb'_0)) & \xrightarrow{\quad - \circ c'_1 \quad} & \pi_0^{(n-1)}(\mathbb{C}_1(Gb_0, Hz'_0)) \\
 \uparrow \scriptstyle \pi_0^{(n-1)} R_1 & & \uparrow \scriptstyle \sigma & & \downarrow \scriptstyle c_1 \circ - \\
 & & & & \pi_0^{(n-1)}(\mathbb{C}_1(Hz_0, Hz'_0))
 \end{array}$$

This shows that $L_1^{p_0, p'_0}$ is itself a comparison between π_0 of an h -pullback and a h -pullback of a π_0 of a diagram (of $(n-1)$ -groupoids), hence it is h -surjective by induction hypothesis.

In conclusion we have shown that $L = \langle L_0, L_1^-, - \rangle$ is h -surjective. \square

Lemma 4.36. *If the n -functor $L : \mathbb{A} \rightarrow \mathbb{K}$ is h -surjective, then also $\pi_0^{(n)}(L)$ is h -surjective.*

Proof. By induction on n .

$n = 1$

Let L be a h -surjective functor between groupoids, *i.e.* L is full and essentially surjective on objects. Therefore for an element $[k_0]_{\sim} \in \pi'_0 \mathbb{K}$ there exists a pair

$$(a_0, k_1 : La_0 \rightarrow k_0)$$

Hence

$$(\pi'_0 L)([a_0]_{\sim}) = [La_0]_{\sim} = [k_0]_{\sim} \quad \text{in } \pi'_0 \mathbb{K}.$$

$n = 2$

Let L be a h -surjective morphism between 2-groupoids, *i.e.*

1. for any k_0 there exist $(a_0, k_1 : La_0 \rightarrow k_0)$.
2. for any pair a_0, a'_0 ,

$$L_1^{a_0, a'_0} : \mathbb{A}_1(a_0, a'_0) \rightarrow \mathbb{K}_1(La_0, La'_0)$$

is h -surjective.

Since $[\pi''_0 L]_0 = L_0$, for any k_0 one has $[k_1]_{\sim} : La_0 \rightarrow k_0$, and this proves the first condition.

Moreover, once we fix a pair a_0, a'_0 , by definition one has $[\pi''_0 L]_1^{a_0, a'_0} = \pi'_0(L_1^{a_0, a'_0})$. Hence it is h -surjective by previous case.

$n > 2$

More generally, a morphism L of n -groupoids is h -surjective when conditions 1. and 2. above both hold. Since $[\pi_0^{(n)} L]_0 = L_0$ and $[\pi_0^{(n)} (\mathbb{K})]_1 = \pi_0^{(n-1)}(\mathbb{K}_1)$, condition 1. for L and for $\pi_0^{(n)} L$ is the same, hence it holds.

Moreover, whence we fix a pair a_0, a'_0 , by definition one has $[\pi_0^{(n)} L]_1^{a_0, a'_0} = \pi_0^{(n-1)}(L_1^{a_0, a'_0})$. Hence it is h -surjective by induction hypothesis. \square

Finally we are ready to state and prove the following important

Theorem 4.37. *Given an exact sequence in $n\mathbf{Gpd}_*$*

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \Downarrow \varepsilon & \swarrow & \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} & \xrightarrow{G} & \mathbb{C} \end{array}$$

the sequence

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \Downarrow \pi_0^{(n)} \varepsilon & \swarrow & \\ \pi_0^{(n)} \mathbb{A} & \xrightarrow{\pi_0^{(n)} F} & \pi_0^{(n)} \mathbb{B} & \xrightarrow{\pi_0^{(n)} G} & \pi_0^{(n)} \mathbb{C} \end{array}$$

is exact in $(n-1)\mathbf{Gpd}_*$

Proof. Let us consider the diagram

$$\begin{array}{ccccc}
 \mathrm{Ker}(\pi_0^{(n)}G) & \xrightarrow{\quad 0 \quad} & & & \\
 \uparrow L' & \searrow & \Downarrow \kappa & & \\
 \pi_0^{(n)}(\mathrm{Ker}G) & \xrightarrow{\quad} & \pi_0^{(n)}\mathbb{B} & \xrightarrow{\pi_0^{(n)}G} & \pi_0^{(n)}\mathbb{C} \\
 \uparrow \pi_0^{(n)}L & \nearrow \pi_0^{(n)}F & & & \\
 \pi_0^{(n)}\mathbb{A} & & & &
 \end{array}$$

L is the comparison in $n\mathbf{Gpd}$, hence h -surjective by hypothesis. Therefore $\pi_0^{(n)}L$ is h -surjective by Lemma 4.36.

L' is the comparison in $(n-1)\mathbf{Gpd}$, h -surjective by Lemma 4.35.

Finally, their composition is again h -surjective by Lemma 4.10, and it is the comparison between $\pi_0^{(n)}\mathbb{A}$ and the kernel of $\pi_0^{(n)}G$ by uniqueness in universal property of h -kernels. \square

4.8 The sesqui-functor π_1

Purpose of this section is to introduce the family of sesqui-functors $\{\pi_1^{(n)}\}_{n \in \mathbb{N}}$ that extends the *isos-of-the-point* functor $\mathbf{Gpd}_* \rightarrow \mathbf{Set}_*$ that assigns to each pointed groupoid the pointed (hom-)set of endo-arrows of the point.

Definition/Proposition 4.38. *For any integer $n > 0$, there exists a sesqui-functor*

$$\pi_1^{(n)} : n\mathbf{Gpd}_* \rightarrow (n-1)\mathbf{Gpd}_*$$

contra-variant on 2-morphisms, according to the following recursive definition.

$$\boxed{n = 1}$$

$\pi_1^{(1)}$ is the functor (= trivial sesqui-functor) $\mathbf{Gpd}_* \rightarrow \mathbf{Set}_*$ that assigns to a pointed groupoid \mathbb{C} the pointed set $\mathbb{C}(*, *)$. It can be considered contra-variant, since 2-morphisms in \mathbf{Set} are equalities.

$$\boxed{n > 1}$$

Let a n -groupoid \mathbb{C} be given. Then $\pi_1^{(n)}\mathbb{C} = \mathbb{C}_1(*, *)$.

Let a morphism of n -groupoids $F : \mathbb{C} \rightarrow \mathbb{D}$ be given. Then $\pi_1^{(n)}F = F_1^{*,*}$.

Of course these assignments give indeed a functor between underlying categories. In fact

$$\pi_1^{(n)}(id_{\mathbb{C}}) = [id_{\mathbb{C}}]_1^{*,*} = id_{\mathbb{C}_1(*,*)} = id_{\pi_1^{(n)}\mathbb{C}}$$

and for every other $G : \mathbb{D} \rightarrow \mathbb{E}$, one has

$$\pi_1^{(n)}(F \bullet^0 G) = [F \bullet^0 G]_1^{*,*} = F_1^{*,*} \bullet^0 G_1^{*,*} = \pi_1^{(n)}(F) \bullet^0 \pi_1^{(n)}(G).$$

Let a 2-morphism $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$ be given. Then we define

$$\pi_1^{(n)}(\alpha) = \alpha_1^{*,*} : \pi_1^{(n)}(G) \Rightarrow \pi_1^{(n)}(F).$$

In fact, since compositions with identities gives identity functors, contra-variance is explained by the following diagram:

$$\begin{array}{ccc} & \mathbb{C}_1(*, *) & \\ F_1^{*,*} \swarrow & & \searrow G_1^{*,*} \\ \mathbb{D}_1(*, *) & \xleftarrow{\alpha_1^{*,*}} & \mathbb{D}_1(*, *) \\ -\circ 1_* \swarrow & & \searrow 1_* \circ - \\ & \mathbb{D}_1(*, *) & \end{array}$$

In order to show that so-defined $\pi_1^{(n)}$ is indeed a sesqui-functor, two facts have to be proved regarding $\pi_1^{(n)}$.

1. it is functorial on hom-categories.
2. it preserves reduced horizontal compositions.

Proof. 1. Suppose we are given

$$\omega : E \Rightarrow F : \mathbb{C} \rightarrow \mathbb{D} \quad \alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$$

in $n\mathbf{Gpd}$. Then

$$\pi_1^{(n)}(\omega \bullet^1 \alpha) = [\omega \bullet^1 \alpha]_1^{*,*} = \alpha_1^{*,*} \bullet^1 \omega_1^{*,*} = \pi_1^{(n)}(\alpha) \bullet^1 \pi_1^{(n)}(\omega)$$

$$\pi_1^{(n)}(id_F) = [id_F]_1^{*,*} = id_{F_1^{*,*}} = id_{\pi_1^{(n)}(F)}$$

2. We prove the statement for reduced left-composition. Suppose 2-morphism α as above, and morphism $N : \mathbb{B} \rightarrow \mathbb{C}$ be given. Then

$$\pi_1^{(n)}(N \bullet_L^0 \alpha) = [N \bullet_L^0 \alpha]_1^{*,*} = N_1^{*,*} \bullet_L^0 \alpha_1^{*,*} = \pi_1^{(n)}(N) \bullet_L^0 \pi_1^{(n)}(\alpha).$$

Finally, concerning reduced right-composition, the proof is similar, as the definition

□

The following proposition is a direct consequence of the definitions. Hence it needs no proof.

Proposition 4.39. *Sesqui-functor $\pi_1^{(n)}$ commutes with finite products and preserves equivalences.*

Remark 4.40. The definition of $\pi_1^{(n)}$ given here makes it of difficult use w.r.t. the inductive setting developed so far, where everything is given as a pair, where the first component lives in **Set**, the second in $(n-1)\mathbf{Cat}$ (or $(n-1)\mathbf{Gpd}$). This motives a further search for a different (but equivalent) definition of $\pi_1^{(n)}$, see *Corollary 6.15*. Thereafter it will also be shown that $\pi_1^{(n)}$ preserves exactness, as a consequence of universality of its definition, although this could be proved here directly.

Chapter 5

3-Morphisms of n -categories

5.1 What structure for $n\text{Cat}$?

So far we have shown that n -categories organizes naturally into a sesqui-category, *ditto* for n -groupoids. This gives a setting to deal not only with n -categories and n -functors, but also with their 2-morphisms, namely lax- n -transformations.

Yet the necessity of introducing 3-morphisms (lax- n -modifications) takes us out of that comfortable setting, into the unknown territory of sesqui-categorically enriched structures.

Following this suggestion, we have named the new setting *sesqui²-category*. This notion is closely related with that of **Gray**-category [Gra76, Gra74] (or 3D-Tas see [Cra00]) and incorporates a horizontal dimension raising composition of 2-morphisms. In fact the set of axioms which define the former is a subset of those defining the latter.

In order to fully justify the name chosen to denote such a structure, it would be interesting to investigate explicitly the enrichment that generates this notion from that of sesqui-category.

What we present here is a treatable inductive approach, comprehensive of a useful characterization given in *Theorem 5.3*.

Definition 5.1. A (small) sesqui²-category \mathcal{C} consists of:

- A 3-truncated reflexive globular set \mathcal{C}_\bullet :

$$\begin{array}{ccccc} \mathcal{C}_3 & \xrightarrow{d_2} & \mathcal{C}_2 & \xrightarrow{d_1} & \mathcal{C}_1 & \xrightarrow{d_0} & \mathcal{C}_0 \\ & \xleftarrow{e_2} & & \xleftarrow{e_1} & & \xleftarrow{e_0} & \\ & \xrightarrow{c_2} & & \xrightarrow{c_1} & & \xrightarrow{c_0} & \end{array}$$

with operations

$$\bullet^m : \mathcal{C}_p \times_{c_m \times d_m} \mathcal{C}_q \rightarrow \mathcal{C}_{p+q-m-1}, \quad m < \min(p, q)$$

such that the following axioms hold:

(i) For every pair $\mathbb{C}, \mathbb{D} \in \mathcal{C}_0$, the localization $\mathcal{C}(\mathbb{C}, \mathbb{D})$ is a sesqui-category, with

- object are F, G , etc. $\in \mathcal{C}_1(\mathbb{C}, \mathbb{D})$
- for any pair of objects F, G , 1-cells are α, β , etc. $\in \mathcal{C}_2(F, G)$
- for any pair of 1-cells $\alpha, \gamma : F \rightarrow G$, 2-cells are Λ, Σ , etc. $\in \mathcal{C}_3(\alpha, \beta)$

k -compositions are restrictions of \bullet^{k+1} -compositions:

- 0-composition of 1-cells of $\mathcal{C}(\mathbb{C}, \mathbb{D})$

$$\diamond^0 := \bullet^1 : \mathcal{C}_2 \times_{c_1 \times d_1} \mathcal{C}_2 \rightarrow \mathcal{C}_2$$

- left/right reduced 0-compositions of 1-cell with a 2-cell of $\mathcal{C}(\mathbb{C}, \mathbb{D})$

$$\diamond_L^0 := \bullet^1 : \mathcal{C}_2 \times_{c_1 \times d_1} \mathcal{C}_3 \rightarrow \mathcal{C}_3$$

$$\diamond_R^0 := \bullet^1 : \mathcal{C}_3 \times_{c_1 \times d_1} \mathcal{C}_2 \rightarrow \mathcal{C}_3$$

- 1-compositions of 2-cells of $\mathcal{C}(\mathbb{C}, \mathbb{D})$

$$\diamond^1 := \bullet^2 : \mathcal{C}_3 \times_{c_2 \times d_2} \mathcal{C}_3 \rightarrow \mathcal{C}_3$$

(ii) For every morphism $F : \mathbb{C} \rightarrow \mathbb{D}$ and objects \mathbb{B}, \mathbb{E} of \mathcal{C}

$$- \bullet^0 F : \mathcal{C}(\mathbb{B}, \mathbb{C}) \rightarrow \mathcal{C}(\mathbb{B}, \mathbb{D})$$

$$F \bullet^0 - : \mathcal{C}(\mathbb{D}, \mathbb{E}) \rightarrow \mathcal{C}(\mathbb{C}, \mathbb{E})$$

are sesqui-functors.

(iii) For every object \mathbb{C} and objects \mathbb{B}, \mathbb{D} of \mathcal{C} , if we denote $id_{\mathbb{C}} = e_0(\mathbb{C})$,

$$- \bullet^0 id_{\mathbb{C}} : \mathcal{C}(\mathbb{B}, \mathbb{C}) \rightarrow \mathcal{C}(\mathbb{B}, \mathbb{C})$$

$$id_{\mathbb{C}} \bullet^0 - : \mathcal{C}(\mathbb{C}, \mathbb{D}) \rightarrow \mathcal{C}(\mathbb{C}, \mathbb{D})$$

are identity sesqui-functors.

(iv) (naturality axioms)

For every pair of 0-composable 2-morphisms $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$ and $\beta : H \Rightarrow K : \mathbb{D} \rightarrow \mathbb{E}$

(a)

$$\alpha \bullet^0 \beta : (F \bullet^0 \beta) \bullet^1 (\alpha \bullet^0 K) \rightarrow (\alpha \bullet_0 H) \bullet^1 (G \bullet^0 \beta)$$

For every 2-morphisms $\varepsilon : L \Rightarrow M : \mathbb{B} \rightarrow \mathbb{C}$ and $\beta : H \Rightarrow K : \mathbb{D} \rightarrow \mathbb{E}$, and for every 3-morphism $\Lambda : \alpha \Rightarrow \omega : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$

(b)

$$(\alpha \bullet^0 \beta) \bullet^2 \left((\Lambda \bullet^0 H) \bullet^1 (G \bullet^0 \beta) \right) = \left((F \bullet^0 \beta) \bullet^1 (\Lambda \bullet^0 K) \right) \bullet^2 (\omega \bullet^0 \beta)$$

and

(c)

$$\left((L \bullet^0 \Lambda) \bullet^1 (\varepsilon \bullet^0 G) \right) \bullet^2 (\varepsilon \bullet^0 \omega) = (\varepsilon \bullet^0 \alpha) \bullet^2 \left((\varepsilon \bullet^0 F) \bullet^1 (M \bullet^0 \Lambda) \right)$$

(v) (functoriality axioms)

For every 2-morphisms $\omega : D \Rightarrow E : \mathbb{B} \rightarrow \mathbb{C}$ and $\gamma : H \Rightarrow L : \mathbb{D} \rightarrow \mathbb{E}$ and every pair of 1-composable 2-morphisms $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$ and $\beta : G \Rightarrow H : \mathbb{C} \rightarrow \mathbb{D}$

(a)

$$(\alpha \bullet^1 \beta) \bullet^0 \gamma = \left((\alpha \bullet^0 \gamma) \bullet^1 (\beta \bullet^0 L) \right) \bullet^2 \left((\alpha \bullet^0 K) \bullet^1 (\beta \bullet^0 \gamma) \right)$$

and

(b)

$$\omega \bullet^0 (\alpha \bullet^1 \beta) = \left((\omega \bullet^0 \alpha) \bullet^1 (E \bullet^0 \beta) \right) \bullet^2 \left((D \bullet^0 \alpha) \bullet^1 (\omega \bullet^0 \beta) \right)$$

(vi) (associativity axiom)

For every 0-composable triple $x \in [\mathcal{C}(\mathbb{B}, \mathbb{C})]_p$, $y \in [\mathcal{C}(\mathbb{C}, \mathbb{D})]_q$ and $z \in [\mathcal{C}(\mathbb{D}, \mathbb{E})]_r$, with $p + q + r \leq 2$

$$(x \bullet^0 y) \bullet^0 z = x \bullet^0 (y \bullet^0 z)$$

(vii) (identity axioms)

For morphisms $E : \mathbb{B} \rightarrow \mathbb{C}$ and $H : \mathbb{D} \rightarrow \mathbb{E}$, and 2-morphism $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$,

$$id_F \bullet^0 \alpha = id_{F \bullet^0 \alpha}, \quad \alpha \bullet^0 id_G = id_{\alpha \bullet^0 G}$$

Remark 5.2. 1. Axiom (iv)(c) is better understood when visualized as in the following diagram (same notation)

$$\begin{array}{ccc} & \omega \bullet^0 H & \\ & \curvearrowright & \\ F \bullet^0 H & \xrightarrow{\alpha \bullet^0 H} & G \bullet^0 H \\ \downarrow F \bullet^0 \beta & \nearrow \alpha \bullet^0 \beta & \downarrow G \bullet^0 \beta \\ F \bullet^0 K & \xrightarrow{\alpha \bullet^0 K} & G \bullet^0 K \end{array} = \begin{array}{ccc} F \bullet^0 H & \xrightarrow{\omega \bullet^0 H} & G \bullet^0 H \\ \downarrow F \bullet^0 \beta & \nearrow \omega \bullet^0 \beta & \downarrow G \bullet^0 \beta \\ F \bullet^0 K & \xrightarrow{\omega \bullet^0 K} & G \bullet^0 K \\ \uparrow \Lambda \bullet^0 K & \nwarrow \alpha \bullet^0 K & \end{array}$$

The same can be claimed for axiom (iv)(b).

2. Axiom (v)(a) is better understood when visualized as in the following diagram (same notation)

$$\begin{array}{ccc} F \bullet^0 K & \xrightarrow{(\alpha \bullet^1 \beta) \bullet^0 K} & H \bullet^0 K \\ \downarrow F \bullet^0 \gamma & \nearrow (\alpha \bullet^1 \beta) \bullet^0 \gamma & \downarrow H \bullet^0 \gamma \\ F \bullet^0 L & \xrightarrow{(\alpha \bullet^1 \beta) \bullet^0 L} & H \bullet^0 L \end{array} = \begin{array}{ccccc} F \bullet^0 K & \xrightarrow{\alpha \bullet^0 K} & G \bullet^0 K & \xrightarrow{\beta \bullet^0 K} & H \bullet^0 K \\ \downarrow F \bullet^0 \gamma & \nearrow \alpha \bullet^0 \gamma & \downarrow G \bullet^0 \gamma & \nearrow \beta \bullet^0 \gamma & \downarrow H \bullet^0 \gamma \\ F \bullet^0 L & \xrightarrow{\alpha \bullet^0 L} & G \bullet^0 L & \xrightarrow{\beta \bullet^0 L} & H \bullet^0 L \end{array}$$

The same can be claimed for axiom (v)(b).

Theorem 5.3. Let \mathcal{C}_\bullet be a 3-truncated reflexive globular set. Then the following two statements are equivalent.

1. \mathcal{C} is a (small) sesqui²-category
2. Axioms (i), (ii) and (iii) of Definition 5.1 hold, moreover

(viii) The 2-truncation $\mathcal{C}_2 \xrightleftharpoons[c_1]{d_1} \mathcal{C}_1 \xrightleftharpoons[c_0]{d_0} \mathcal{C}_0$ of \mathcal{C}_\bullet is a sesqui-category.

(ix) For every 2-morphism $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$ and objects \mathbb{B}, \mathbb{E} of \mathcal{C}

$$- \bullet^0 \alpha : - \bullet^0 F \Rightarrow - \bullet^0 G : \mathcal{C}(\mathbb{B}, \mathbb{C}) \rightarrow \mathcal{C}(\mathbb{B}, \mathbb{D})$$

$$\alpha \bullet^0 - : F \bullet^0 - \Rightarrow G \bullet^0 - : \mathcal{C}(\mathbb{D}, \mathbb{E}) \rightarrow \mathcal{C}(\mathbb{C}, \mathbb{E})$$

are lax natural transformations of sesqui-functors.

(x) For every morphism $F : \mathbb{C} \rightarrow \mathbb{D}$ of \mathcal{C}

$$- \bullet^0 id_F - \bullet^0 F \Rightarrow - \bullet^0 F$$

$$id_F \bullet^0 - : F \bullet^0 - \Rightarrow F \bullet^0 -$$

are identical natural transformations.

(xi) (reduced associativity axiom)

For every 0-composable triple $x \in [\mathcal{C}(\mathbb{B}, \mathbb{C})]_p$, $y \in [\mathcal{C}(\mathbb{C}, \mathbb{D})]_q$ and $z \in [\mathcal{C}(\mathbb{D}, \mathbb{E})]_r$, with $p + q + r = 2$

$$(x \bullet^0 y) \bullet^0 z = x \bullet^0 (y \bullet^0 z)$$

i.e. for 3-morphism Λ , 2-morphisms α, β and morphisms F, G of \mathcal{C} , the following equations hold, when composites exist:

$$\begin{aligned} (\Lambda \bullet^0 F) \bullet^0 G &= \Lambda \bullet^0 (F \bullet^0 G) & (\alpha \bullet^0 \beta) \bullet^0 F &= \alpha \bullet^0 (\beta \bullet^0 F) \\ (F \bullet^0 \Lambda) \bullet^0 F &= F \bullet^0 (\Lambda \bullet^0 G) & (\alpha \bullet^0 F) \bullet^0 \beta &= \alpha \bullet^0 (F \bullet^0 \beta) \\ (\Lambda \bullet^0 F) \bullet^0 G &= \Lambda \bullet^0 (F \bullet^0 G) & (F \bullet^0 \alpha) \bullet^0 \beta &= F \bullet^0 (\alpha \bullet^0 \beta) \end{aligned}$$

Proof. First we prove that 1. implies 2..

Condition (viii) is equivalent to satisfying properties (L1) to (L4), (R1) to (R4) and (LR5) of *Proposition 2.2*. Now, (L1) and (R1) hold by (iii), (L2) and (R2) by (iv), (L3), (R3), (L4) and (R4) by (ii), (LR5) by (vi).

Condition (ix) holds. In fact let us recall *Definition 2.7*. Assignment on objects (=1-cells) is given by 0-composition, naturality by (iv) and functoriality by (v) (compositions) and (vii) (units).

Condition (x) holds too. In fact this is implied by (ix) above and (vii).

Finally (xi) is a subset of (vi).

Conversely we prove that 2. implies 1..

Conditions (iv) and (v) hold by (ix).

Condition (vi) holds by (xi) for the cases $p + q + r = 2$. What is still to prove is the case $p + q + r = 0$ and the case $p + q + r = 1$, that are given by (viii).

Finally (vii) is a consequence of (ix) and (x). \square

Remark 5.4. Notice that the characterization given by *Theorem 5.3* is somehow redundant. Nevertheless its usefulness is that it makes available practical rules in order to deal with calculations in a sesqui²-categorical environment.

5.2 Lax n -modification

Purpose of the rest of the chapter is to give a proof of the following

Theorem 5.5. *The sesqui-category $n\mathbf{Cat}$, endowed with 3-morphism, their compositions, whiskering and dimension raising 0-composition of 2-morphisms is a sesqui²-category.*

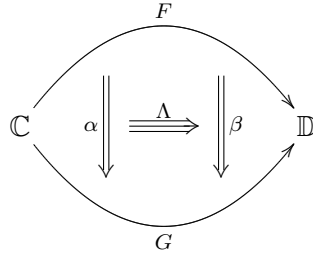
This is done by means of the characterization given in *Theorem 5.3*.

As usual the approach is genuinely inductive, starting with the well known definition of a *modification* in \mathbf{Cat} [Bor94].

Hence suppose given an integer $n > 1$. Let us consider the following situation in $n\mathbf{Cat}$:

$$\alpha, \beta : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$$

A lax n -modification $\Lambda : \alpha \Longrightarrow \beta$



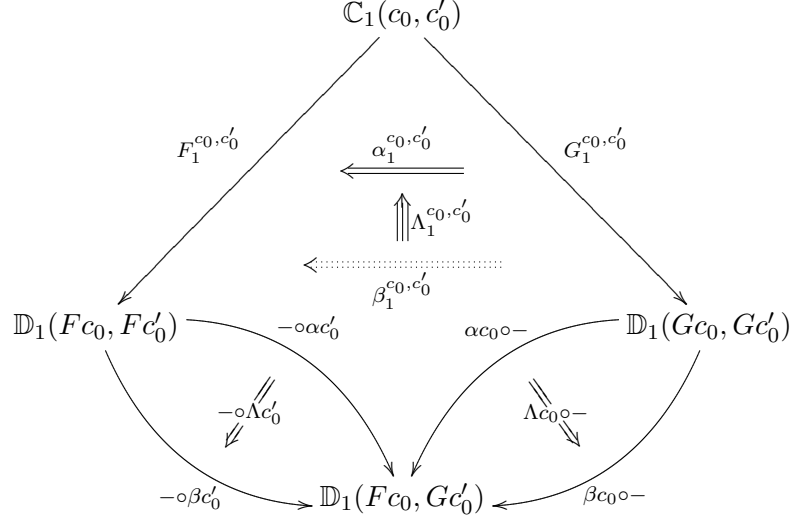
is a pair (Λ_0, Λ_1) , where

- $\Lambda_0 : \mathbb{C}_0 \longrightarrow \coprod_{c_0 \in \mathbb{C}_0} [\mathbb{D}_2(\alpha_0(c_0), \beta_0(c_0))]_0$ is a map such that, for every c_0 in \mathbb{C}_0 , $\Lambda_0(c_0) : \alpha_0(c_0) \Longrightarrow \beta_0(c_0)$.

Let us point out that subscript “0” is sometimes omitted (as in $\alpha(c_0)$), or c_0 is itself subscripted (as in α_{c_0}).

- (n -naturality) for every pair of objects c_0, c'_0 of \mathbb{C} , a 3-morphism of

$(n - 1)$ categories that *fills* the following diagram:



i.e.

$$\begin{array}{ccc}
 G_1^{c_0, c'_0} \bullet^0 (- \circ \alpha c'_0) & \xrightarrow{\alpha_1^{c_0, c'_0}} & F_1^{c_0, c'_0} \bullet^0 (- \circ \alpha c'_0) \\
 \downarrow id \bullet_0 (\Lambda c_0 \circ -) & \nearrow \Lambda_1^{c_0, c'_0} & \downarrow id \bullet_0 (- \circ \Lambda c_0) \\
 G_1^{c_0, c'_0} \bullet^0 (\beta c_0 \circ -) & \xrightarrow{\beta_1^{c_0, c'_0}} & F_1^{c_0, c'_0} \bullet^0 (- \circ \beta c'_0)
 \end{array}$$

These data must obey to *functoriality* axioms described by the following equations of 3-diagrams in $(n-1)\mathbf{Cat}$:

- (*functoriality w.r.t. 0-composition*) for every triple c_0, c'_0, c''_0 of objects of \mathbb{C}

$$\begin{array}{ccc}
\mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c''_0) & \xlongequal{\quad} & \mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c''_0) \\
\downarrow \scriptstyle id \times \beta_1^{c'_0, c''_0} \quad \swarrow \scriptstyle id \times \alpha_1^{c'_0, c''_0} & & \downarrow \scriptstyle \beta_1^{c_0, c'_0} \times id \quad \swarrow \scriptstyle \alpha_1^{c_0, c'_0} \times id \\
\mathbb{C}_1(c_0, c'_0) \times \mathbb{D}_1(Fc'_0, Gc''_0) & & \mathbb{D}_1(Fc_0, Gc'_0) \times \mathbb{C}_1(c'_0, c''_0) \\
\downarrow \scriptstyle F(-) \circ - & & \downarrow \scriptstyle - \circ G(-) \\
& \mathbb{D}_1(Fc_0, Gc''_0) & \\
& = & \\
& \mathbb{C}_1(c_0, c'_0) \times \mathbb{C}_1(c'_0, c''_0) & \\
& \downarrow \scriptstyle \circ & \\
& \mathbb{C}_1(c_0, c''_0) & \\
& \downarrow & \\
& \mathbb{D}_1(Fc_0, Gc''_0) &
\end{array}
\tag{5.1}$$

$\mathbb{C}_1(c_0, c''_0)$

$\mathbb{D}_1(Fc_0, Gc''_0)$

namely:

$$(\Lambda_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0}) \bullet^2 (F_1^{c_0, c'_0} \circ \Lambda_1^{c'_0, c''_0}) = (- \circ -) \bullet^0 \Lambda_1^{c_0, c''_0}$$

where the 2-dimensional intersection is the 2-morphism $F(-) \circ \Lambda_{c'_0} \circ G(-)$.

- (functoriality w.r.t. units) for every object c_0 of \mathbb{C}

$$(5.2)$$

namely:

$$u(c_0) \bullet^0 \Lambda_1^{c_0, c_0} = Id_{[\Lambda_{c_0}]}$$

We write $[\Lambda_{c_0}]$ for the constant 2-morphism given by Λ_{c_0} ; in this case it is between constant morphisms:

$$[\Lambda_{c_0}] : [\alpha_{c_0}] \Rightarrow [\beta_{c_0}] : \mathbb{I}_{(n-1)} \rightarrow \mathbb{D}_1(Fc_0, Gc_0)$$

Notice that both functoriality axioms for 3-morphisms reduce to those for 2-morphisms, when we consider only identity 3-morphisms (i.e. 2-morphisms *considered as* 3-morphisms).

In the same way functoriality axioms for 2-morphisms reduce to those for 1-morphisms, when we consider only identity 3-morphisms (i.e. 2-morphisms *considered as* 3-morphisms).

5.3 $n\mathbf{Cat}(\mathbb{C}, \mathbb{D})$: the underlying category

Here and in the following three sections we consider n -categories \mathbb{C} and \mathbb{D} be given. We consider a sesqui-category structure over the category $n\mathbf{Cat}(\mathbb{C}, \mathbb{D})$. As we did in defining the sesqui-category $n\mathbf{Cat}$, we start by showing the underlying category structure. This has been already detailed in section 3.3, hence it suffices to recall that:

- objects of $[n\mathbf{Cat}(\mathbb{C}, \mathbb{D})]$ are n -functors $\mathbb{C} \rightarrow \mathbb{D}$;

- arrows of $[n\mathbf{Cat}(\mathbb{C}, \mathbb{D})]$ n -lax transformation between them.

Composition is 2-morphisms 1-composition, obvious units.

5.4 $n\mathbf{Cat}(\mathbb{C}, \mathbb{D})$: the hom-categories

Let us fix n -functors $F, G : \mathbb{C} \rightarrow \mathbb{D}$. We have to define categories $(n\mathbf{Cat}(\mathbb{C}, \mathbb{D}))(F, G)$, or more simply $n\mathbf{Cat}(F, G)$.

- Objects of $n\mathbf{Cat}(F, G)$ are 2-morphisms $\alpha : F \Rightarrow G$;
- Arrows $\alpha \rightarrow \beta$ are 3-morphisms of n -categories.

5.4.1 Composition

For 3-morphisms $\Lambda = (\Lambda_0, \Lambda_1^-, -) : \alpha \rightarrow \beta$ and $\Sigma = (\Sigma_0, \Sigma_1^-, -) : \beta \rightarrow \gamma$ their 2-composition $\Lambda \bullet^2 \Sigma : \alpha \rightarrow \gamma$ is given by the following data:

- (on objects)

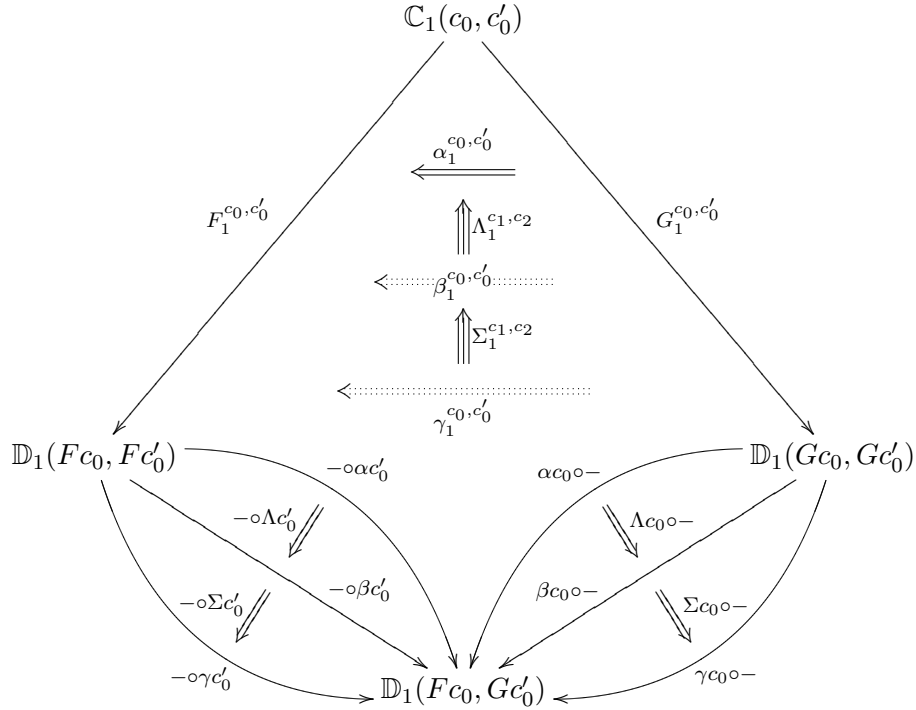
$$[\Lambda \bullet^2 \Sigma]_0 : c_0 \mapsto \Lambda c_0 \circ^1 \Sigma c_0$$

i.r.

$$\alpha c_0 \xRightarrow{\Lambda c_0} \beta c_0 \xRightarrow{\Sigma c_0} \gamma c_0$$

- (on homs) For chosen objects c_0, c'_0 one has

$$[\Lambda \bullet^2 \Sigma]_1^{c_0, c'_0} = \left((G_1^{c_0, c'_0} \bullet^0 (\Lambda c_0 \circ -)) \bullet^1 \Sigma_1^{c_0, c'_0} \right) \bullet^2 \left(\Lambda_1^{c_0, c'_0} \bullet^1 (F_1^{c_0, c'_0} \bullet^0 (- \circ \Sigma c'_0)) \right)$$



We can represent this also as a 2-dimensional pasting, sometimes useful in proofs:

$$\begin{array}{ccccc}
 G_1^{c_0, c'_0} \bullet^0 (\alpha c_0 \circ -) & \xrightarrow{id \bullet^0 (\Lambda c_0 \circ -)} & G_1^{c_0, c'_0} \bullet^0 (\beta c_0 \circ -) & \xrightarrow{id \bullet^0 (\Sigma c_0 \circ -)} & G_1^{c_0, c'_0} \bullet^0 (\gamma c_0 \circ -) \\
 \downarrow \alpha_1^{c_0, c'_0} & \swarrow \Lambda_1^{c_0, c'_0} & \downarrow \beta_1^{c_0, c'_0} & \swarrow \Sigma_1^{c_0, c'_0} & \downarrow \gamma_1^{c_0, c'_0} \\
 F_1^{c_0, c'_0} \bullet^0 (- \circ \alpha c'_0) & \xrightarrow{id \bullet^0 (- \circ \Lambda c'_0)} & F_1^{c_0, c'_0} \bullet^0 (- \circ \beta c'_0) & \xrightarrow{id \bullet^0 (- \circ \Sigma c'_0)} & F_1^{c_0, c'_0} \bullet^0 (- \circ \gamma c'_0)
 \end{array}$$

Notice that

$$(- \circ \Lambda c'_0) \bullet^1 (- \circ \Sigma c'_0) = - \circ (\Lambda c'_0 \circ \Sigma c'_0) = - \circ [\Lambda \bullet^2 \Sigma] c'_0$$

$$(\Lambda c_0 \circ -) \bullet^1 (\Sigma c_0 \circ -) = (\Lambda c_0 \circ \Sigma c_0) \circ - = [\Lambda \bullet^2 \Sigma] c_0 \circ -$$

These data form indeed a 3-morphism. In fact let us consider the following diagram for every triple of objects c_0, c'_0, c''_0 of \mathbb{C}

$$\begin{array}{ccc}
 \alpha c_0 \circ G_1^{c_0, c'_0}(-) \circ G_1^{c'_0, c''_0}(-) & \xrightarrow{[\Lambda \bullet^2 \Sigma] c_0 \circ id} & \gamma c_0 \circ G_1^{c_0, c'_0}(-) \circ G_1^{c'_0, c''_0}(-) \\
 \downarrow \alpha_1^{c_0, c'_0} \circ id & \swarrow [\Lambda \bullet^2 \Sigma]_1^{c_0, c'_0} \circ id & \downarrow \gamma_1^{c_0, c'_0} \circ id \\
 F_1^{c_0, c'_0}(-) \circ \alpha c'_0 \circ G_1^{c'_0, c''_0}(-) & \xrightarrow{id \circ [\Lambda \bullet^2 \Sigma] c'_0 \circ id} & F_1^{c_0, c'_0}(-) \circ \gamma c'_0 \circ G_1^{c'_0, c''_0}(-) \\
 \downarrow id \circ \alpha_1^{c'_0, c''_0} & \swarrow id \circ [\Lambda \bullet^2 \Sigma]_1^{c'_0, c''_0} & \downarrow id \circ \gamma_1^{c'_0, c''_0} \\
 F_1^{c_0, c'_0}(-) \circ F_1^{c'_0, c''_0}(-) \circ \alpha c''_0 & \xrightarrow{id \circ [\Lambda \bullet^2 \Sigma] c''_0} & F_1^{c_0, c'_0}(-) \circ F_1^{c'_0, c''_0}(-) \circ \gamma c''_0
 \end{array}$$

by definition of $[\Lambda \bullet^2 \Sigma]_1^{-, -}$ one has

$$\begin{array}{ccccc}
\alpha_{c_0} \circ G_1^{c_0, c'_0}(-) \circ G_1^{c'_0, c''_0}(-) & \xrightarrow{\Lambda_{c_0} \circ id} & \beta_{c_0} \circ G_1^{c_0, c'_0}(-) \circ G_1^{c'_0, c''_0}(-) & \xrightarrow{\Sigma_{c_0} \circ id} & \gamma_{c_0} \circ G_1^{c_0, c'_0}(-) \circ G_1^{c'_0, c''_0}(-) \\
\downarrow \alpha_1^{c_0, c'_0} \circ id & \swarrow \Lambda_1^{c_0, c'_0} \circ id & \downarrow \beta_1^{c_0, c'_0} \circ id & \swarrow \Sigma_1^{c_0, c'_0} \circ id & \downarrow \gamma_1^{c_0, c'_0} \circ id \\
F_1^{c_0, c'_0}(-) \circ \alpha_{c'_0} \circ G_1^{c'_0, c''_0}(-) & \xrightarrow{id \circ \Lambda_{c'_0} \circ id} & F_1^{c_0, c'_0}(-) \circ \beta_{c'_0} \circ G_1^{c'_0, c''_0}(-) & \xrightarrow{id \circ \Sigma_{c'_0} \circ id} & F_1^{c_0, c'_0}(-) \circ \gamma_{c'_0} \circ G_1^{c'_0, c''_0}(-) \\
\downarrow id \circ \alpha_1^{c'_0, c''_0} & \swarrow id \circ \Lambda_1^{c'_0, c''_0} & \downarrow id \circ \beta_1^{c'_0, c''_0} & \swarrow id \circ \Sigma_1^{c'_0, c''_0} & \downarrow id \circ \gamma_1^{c'_0, c''_0} \\
F_1^{c_0, c'_0}(-) \circ F_1^{c'_0, c''_0}(-) \circ \alpha_{c''_0} & \xrightarrow{id \circ \Lambda_{c''_0}} & F_1^{c_0, c'_0}(-) \circ F_1^{c'_0, c''_0}(-) \circ \beta_{c''_0} & \xrightarrow{id \circ \Sigma_{c''_0}} & F_1^{c_0, c'_0}(-) \circ F_1^{c'_0, c''_0}(-) \circ \gamma_{c''_0}
\end{array} \tag{5.3}$$

This diagram is unambiguous because interchange holds on separate components of product (*product interchange* in dimension $n - 1$, with intersection the constant $[\beta_{c'_0}]$). Hence we get

$$\begin{array}{ccccc}
\alpha_{c_0} \circ G_1^{c_0, c''_0}(- \circ -) & \xrightarrow{\Lambda_{c_0} \circ id} & \beta_{c_0} \circ G_1^{c_0, c''_0}(- \circ -) & \xrightarrow{\Sigma_{c_0} \circ id} & \gamma_{c_0} \circ G_1^{c_0, c''_0}(- \circ -) \\
\downarrow \alpha_1^{c_0, c''_0} & \swarrow \Lambda_1^{c_0, c''_0} & \downarrow \beta_1^{c_0, c''_0} & \swarrow \Sigma_1^{c_0, c''_0} & \downarrow \gamma_1^{c_0, c''_0} \\
F_1^{c_0, c''_0}(- \circ -) \circ \alpha_{c''_0} & \xrightarrow{id \circ \Lambda_{c''_0}} & F_1^{c_0, c''_0}(- \circ -) \circ \alpha_{c''_0} & \xrightarrow{id \circ \Sigma_{c''_0}} & F_1^{c_0, c''_0}(- \circ -) \circ \alpha_{c''_0}
\end{array}$$

More simply for any object c_0 of \mathbb{C} one has

$$\begin{array}{ccccc}
\alpha_{c_0} \circ G_1^{c_0, c_0}(u(c_0)) & \xrightarrow{\Lambda_{c_0} \circ id} & \beta_{c_0} \circ G_1^{c_0, c_0}(u(c_0)) & \xrightarrow{\Sigma_{c_0} \circ id} & \gamma_{c_0} \circ G_1^{c_0, c_0}(u(c_0)) \\
\downarrow \alpha_1^{c_0, c_0} & \swarrow \Lambda_1^{c_0, c_0} & \downarrow \beta_1^{c_0, c_0} & \swarrow \Sigma_1^{c_0, c_0} & \downarrow \gamma_1^{c_0, c_0} \\
F_1^{c_0, c_0}(u(c_0)) \circ \alpha_{c_0} & \xrightarrow{id \circ \Lambda_{c_0}} & F_1^{c_0, c_0}(u(c_0)) \circ \alpha_{c_0} & \xrightarrow{id \circ \Sigma_{c_0}} & F_1^{c_0, c_0}(u(c_0)) \circ \alpha_{c_0} \\
& & = & &
\end{array} \tag{5.4}$$

$$\begin{array}{ccccc}
[\alpha_{c_0}] & \xrightarrow{[\Lambda_{c_0}]} & [\beta_{c_0}] & \xrightarrow{[\Sigma_{c_0}]} & [\gamma_{c_0}] \\
\parallel & \swarrow id & \parallel & \swarrow id & \parallel \\
[\alpha_{c_0}] & \xrightarrow{[\Lambda_{c_0}]} & [\beta_{c_0}] & \xrightarrow{[\Sigma_{c_0}]} & [\gamma_{c_0}]
\end{array}$$

where, as usual, square brackets mean *constant*.

5.4.2 Units

For any 2-morphism $\beta : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$ its identity 3-morphisms id_β is given by:

- (on objects)

$$[id_\beta]_0 : c_0 \mapsto id_{\beta c_0}$$

- (on homs) For chosen objects c_0, c'_0 one has

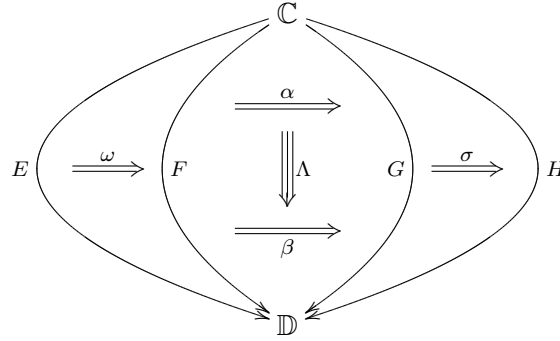
$$[id_\beta]_1^{-,-} = id_{\beta_1^{-,-}}$$

It is immediate to check that above pair is indeed a 3-morphisms.

Similarly associativity and neutral units follows from same properties for 2-cells and from diagrams (5.3) and (5.4) suitably adapted (*adding one more column, for what concerns associativity, trivializing one column, for what concerns units*).

5.5 $n\mathbf{Cat}(\mathbb{C}, \mathbb{D})$: the sesqui-categorical structure

In the this section we will show that hom-categories $n\mathbf{Cat}(\mathbb{C}, \mathbb{D})$ underly a structure of sesqui-categories, with 2-cells provided by 3-morphisms of n-categories. To this end we define reduced left/right 1-composition of a 3-morphism with a 2-morphism, according to the following reference diagram.



5.5.1 Reduced left-composition

The 3-morphism

$$\omega \bullet^1 \Lambda : \omega \bullet^1 \alpha \Rightarrow \omega \bullet^1 \beta : E \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$$

is given by the data below

- (on objects) For an object c_0 of \mathbb{C}

$$[\omega \bullet^1 \Lambda]_0 : c_0 \mapsto \begin{array}{c} Ec_0 \\ \downarrow \omega c_0 \\ Fc_0 \\ \alpha c_0 \begin{array}{c} \xrightarrow{\Lambda c_0} \\ \xleftarrow{\beta c_0} \end{array} \beta c_0 \\ Gc_0 \end{array}$$

- (on homs) For objects c_0, c'_0 of \mathbb{C}

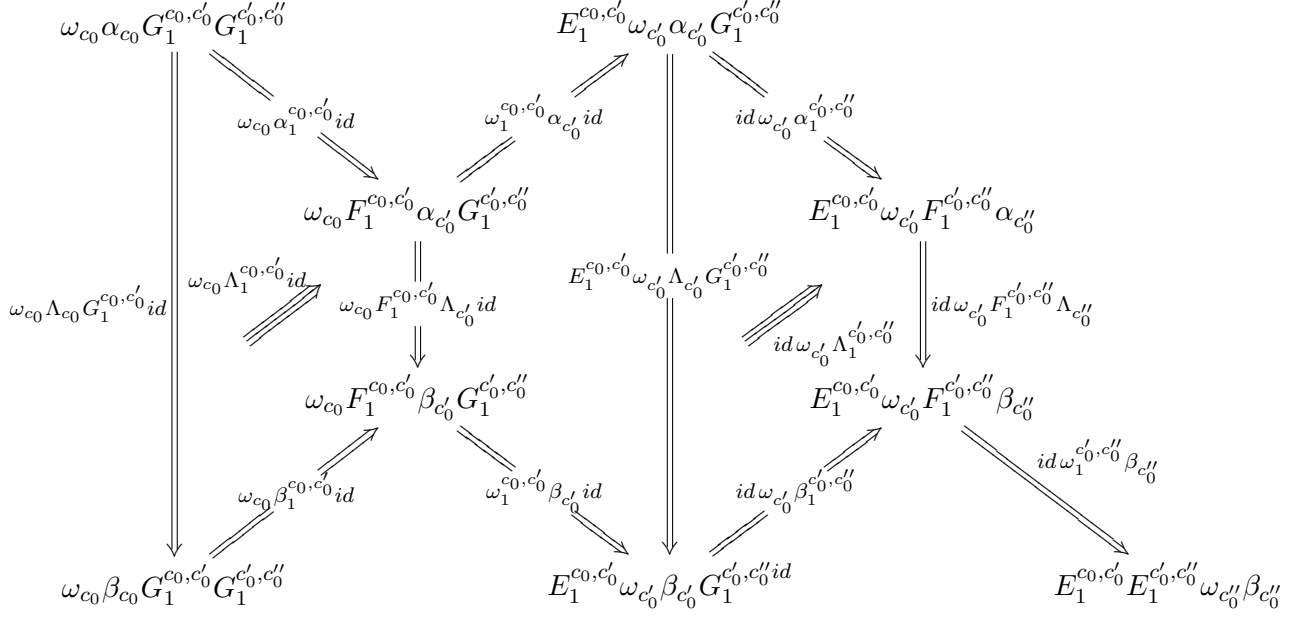
$$\begin{aligned} [\omega \bullet^1 \Lambda]_1^{c_0, c'_0} &= \left(\Lambda_1^{c_0, c'_0} \bullet^0 (\omega c_0 \circ -) \right) \bullet^1 \left(\omega_1^{c_0, c'_0} \bullet^0 (- \circ \beta c'_0) \right) \\ &= (\omega c_0 \circ \Lambda_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \beta c'_0) \end{aligned}$$

$$\begin{array}{ccc} \omega c_0 \circ \alpha c_0 \circ G_1^{c_0, c'_0}(-) & \xrightarrow{[\omega \bullet^1 \alpha]_1^{c_0, c'_0}} & E_1^{c_0, c'_0}(-) \circ \omega c'_0 \circ \alpha c'_0 \\ \parallel & \searrow \omega c_0 \circ \alpha_1^{c_0, c'_0} & \nearrow \omega_1^{c_0, c'_0} \circ \alpha c'_0 \\ & \omega c_0 \circ F_1^{c_0, c'_0}(-) \circ \alpha c'_0 & \\ & \parallel \omega c_0 \circ F_1^{c_0, c'_0}(-) \circ \Lambda c'_0 & \\ & \omega c_0 \circ F_1^{c_0, c'_0}(-) \circ \beta c'_0 & \\ \nearrow \omega c_0 \circ \beta_1^{c_0, c'_0} & \searrow \omega_1^{c_0, c'_0} \circ \beta c'_0 & \\ \omega c_0 \circ \beta c_0 \circ G_1^{c_0, c'_0}(-) & \xrightarrow{[\omega \bullet^1 \beta]_1^{c_0, c'_0}} & E_1^{c_0, c'_0}(-) \circ \omega c'_0 \circ \beta c'_0 \end{array}$$

$\begin{array}{l} [\omega \bullet^1 \Lambda]_{c_0 \circ G_1^{c_0, c'_0}(-)} \\ = \omega c_0 \circ \Lambda c_0 \circ G_1^{c_0, c'_0}(-) \end{array} \quad \begin{array}{l} E_1^{c_0, c'_0}(-) \circ [\omega \bullet^1 \Lambda]_{c'_0} \\ = E_1^{c_0, c'_0}(-) \circ \omega c'_0 \circ \Lambda c'_0 \end{array}$

The pair $\langle [\omega \bullet^1 \Lambda]_0, [\omega \bullet^1 \Lambda]_1^{-, -} \rangle$ forms indeed a 3-morphism of n -categories.

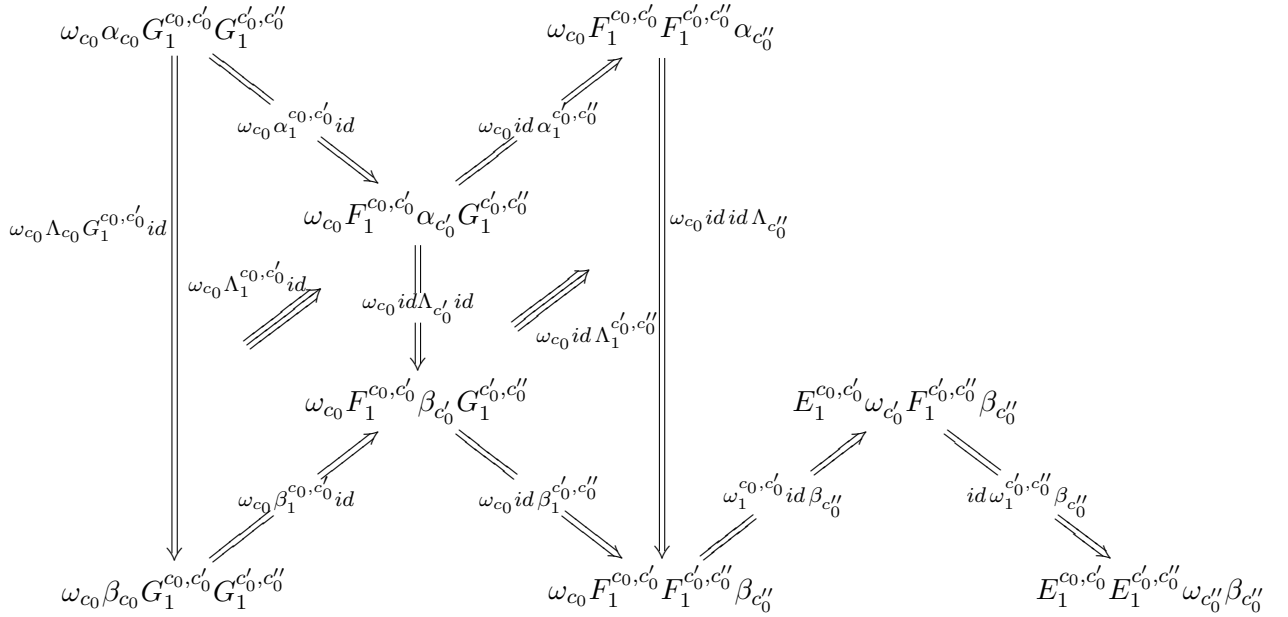
Proof. We have to show that it satisfies composition and unit axioms. Let us begin with composition, and fix a triple c_0, c'_0, c''_0 of \mathbb{C} . Notice that, in order to keep diagrams in the page we denote \circ^0 -composition by juxtaposition, and subscripts for transformations on objects are used.



By product interchange it is clear that

$$\left(\omega_1^{c_0, c'_0} \circ \alpha_{c'_0} \circ id_{G_1^{c'_0, c''_0}} \right) \bullet^1 \left(id_{E_1^{c_0, c'_0}} \circ \omega_{c'_0} \circ \Lambda_1^{c'_0, c''_0} \right) = \left(\omega_{c_0} \circ F_1^{c_0, c'_0} \circ \Lambda_1^{c'_0, c''_0} \right) \bullet^1 \left(\omega_1^{c_0, c'_0} \circ id_{F_1^{c'_0, c''_0}} \circ \beta_{c''_0} \right)$$

and the diagram can be re-drawn



Now let us consider the pasting of the left-hand side of above diagram, and write it equationally:

$$\begin{aligned} & \left[(\Lambda_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0}) \bullet^0 (\omega_{c_0} \circ -) \right] \bullet^1 \left[(F_1^{c_0, c'_0} \circ \beta_1^{c'_0, c''_0}) \bullet^0 (\omega_{c_0} \circ -) \right] \\ & \quad \bullet^2 \\ & \left[(\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0}) \bullet^0 (\omega_{c_0} \circ -) \right] \bullet^1 \left[(F_1^{c_0, c'_0} \circ \Lambda_1^{c'_0, c''_0}) \bullet^0 (\omega_{c_0} \circ -) \right] \end{aligned}$$

By *whiskering interchange property (LRW)* this equals to

$$\begin{aligned} & \left[(\Lambda_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0}) \bullet^1 (F_1^{c_0, c'_0} \circ \beta_1^{c'_0, c''_0}) \right] \bullet^0 (\omega_{c_0} \circ -) \\ & \quad \bullet^2 \\ & \left[(\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0}) \bullet^1 (F_1^{c_0, c'_0} \circ \Lambda_1^{c'_0, c''_0}) \right] \bullet^0 (\omega_{c_0} \circ -) \end{aligned}$$

and by functoriality of right 0-whiskering $(R4)''$

$$\left[\left[(\Lambda_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0}) \bullet^1 (F_1^{c_0, c'_0} \circ \beta_1^{c'_0, c''_0}) \right] \bullet^2 \left[(\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0}) \bullet^1 (F_1^{c_0, c'_0} \circ \Lambda_1^{c'_0, c''_0}) \right] \right] \bullet^0 (\omega_{c_0} \circ -)$$

finally by functoriality w.r.t. composition of 3-morphism Λ

$$\Lambda_1^{c_0, c''_0} \bullet^0 (\omega_{c_0} \circ -) = \omega_{c_0} \circ \Lambda_1^{c_0, c''_0}$$

Concerning composite 2-morphism on the right-hand side, we can apply whiskering properties and composition axiom for 2-morphisms

$$(\omega_1^{c_0, c'_0} \circ F_1^{c_0, c''_0} \circ \beta_{c'_0}) \bullet^1 (E_1^{c_0, c'_0} \circ \omega_1^{c'_0, c''_0} \circ \beta_{c''_0}) = \omega_1^{c_0, c''_0} \circ \beta_{c''_0}$$

and diagram above can be re-drawn as follows

$$\begin{array}{ccccc} \omega_{c_0} \alpha_{c_0} G_1^{c_0, c''_0} & \xrightarrow{\omega_{c_0} \alpha_1^{c_0, c''_0}} & \omega_{c_0} F_1^{c_0, c''_0} \alpha_{c''_0} & & \\ \downarrow \omega_{c_0} \Lambda_{c_0} id & \nearrow \omega_{c_0} \Lambda_1^{c_0, c''_0} & \downarrow \omega_{c_0} id \Lambda_{c''_0} & & \\ \omega_{c_0} \beta_{c_0} G_1^{c_0, c''_0} & \xrightarrow{\omega_{c_0} \beta_1^{c_0, c''_0}} & \omega_{c_0} F_1^{c_0, c''_0} \beta_{c''_0} & \xrightarrow{\omega_1^{c_0, c''_0} \beta_{c''_0}} & E_1^{c_0, c''_0} \omega_{c''_0} \beta_{c''_0} \end{array}$$

and this conclude the proof of composition axiom.

Concerning units, for an object c_0 of \mathbb{C} one has

$$\begin{array}{ccccc}
 \omega_{c_0} \alpha_{c_0} G_1^{c_0, c_0}(u(c_0)) & \xrightarrow{\omega_{c_0} \alpha_1^{c_0, c_0}} & \omega_{c_0} F_1^{c_0, c_0}(u(c_0)) \alpha_{c_0} & & \\
 \downarrow \omega_{c_0} \Lambda_{c_0} id & \nearrow \omega_{c_0} \Lambda_1^{c_0, c_0} & \downarrow \omega_{c_0} id \Lambda_{c_0} & & \\
 \omega_{c_0} \beta_{c_0} G_1^{c_0, c_0}(u(c_0)) & \xrightarrow[\omega_{c_0} \beta_1^{c_0, c_0}]{} & \omega_{c_0} F_1^{c_0, c_0}(u(c_0)) \beta_{c_0} & \xrightarrow[\omega_1^{c_0, c_0} \beta_{c_0}]{} & E_1^{c_0, c_0}(u(c_0)) \omega_{c_0} \beta_{c_0}
 \end{array}$$

then by functoriality w.r.t. units of 3-morphisms of (n-1)categories (and also by functoriality w.r.t. units of 2-morphisms and 1-morphisms) we get

$$\begin{array}{ccccc}
 \omega_{c_0} [\alpha_{c_0}] & \xrightarrow{\omega_{c_0} id} & \omega_{c_0} [\alpha_{c_0}] & & \\
 \downarrow \omega_{c_0} [\Lambda_{c_0}] & \nearrow \omega_{c_0} id & \downarrow \omega_{c_0} [\Lambda_{c_0}] & & \\
 \omega_{c_0} [\beta_{c_0}] & \xrightarrow[\omega_{c_0} id]{} & \omega_{c_0} [\beta_{c_0}] & \xrightarrow[\omega_{c_0} id]{} & \omega_{c_0} [\beta_{c_0}]
 \end{array}$$

hence the result. \square

5.5.2 Reduced right-composition

The 3-morphism

$$\Lambda \bullet^1 \sigma : \alpha \bullet^1 \sigma \Rightarrow \beta \bullet^1 \sigma : F \Rightarrow H : \mathbb{C} \rightarrow \mathbb{D}$$

is given by the data below

- (on objects) For an object c_0 of \mathbb{C}

$$\begin{array}{c}
 [\Lambda \bullet^1 \sigma]_0 : c_0 \mapsto \begin{array}{c} \begin{array}{ccc} Fc_0 & & \\ \alpha_{c_0} \swarrow & \xRightarrow{\Lambda c_0} & \searrow \beta_{c_0} \\ Gc_0 & & \\ \downarrow \sigma_{c_0} & & \\ Hc_0 & & \end{array} \end{array}
 \end{array}$$

- (on homs) For objects c_0, c'_0 of \mathbb{C}

$$\begin{aligned} [\Lambda \bullet^1 \sigma]_1^{c_0, c'_0} &= \left(\sigma_1^{c_0, c'_0} \bullet^0 (\alpha c_0 \circ -) \right) \bullet^1 \left(\Lambda_1^{c_0, c'_0} \bullet^0 (- \circ \sigma c'_0) \right) \\ &= (\alpha c_0 \circ \sigma_1^{c_0, c'_0}) \bullet^1 (\Lambda_1^{c_0, c'_0} \circ \sigma c'_0) \end{aligned}$$

$$\begin{array}{ccc} \alpha c_0 \circ \sigma c_0 \circ H_1^{c_0, c'_0}(-) & \xrightarrow{[\alpha \bullet^1 \sigma]_1^{c_0, c'_0}} & F_1^{c_0, c'_0}(-) \circ \alpha c'_0 \circ \sigma c'_0 \\ & \searrow \alpha c_0 \circ \sigma_1^{c_0, c'_0} & \nearrow \alpha_1^{c_0, c'_0} \circ \sigma c'_0 \\ & \alpha c_0 \circ G_1^{c_0, c'_0}(-) \circ \sigma c'_0 & \\ & \downarrow \Lambda c_0 \circ G_1^{c_0, c'_0} \circ \sigma c'_0 & \nearrow \Lambda_1^{c_0, c'_0} \circ \sigma c'_0 \\ & \beta c_0 \circ G_1^{c_0, c'_0}(-) \circ \sigma c'_0 & \\ & \nwarrow \beta c_0 \circ \sigma_1^{c_0, c'_0} & \searrow \beta_1^{c_0, c'_0} \circ \sigma c'_0 \\ \beta c_0 \circ \sigma c_0 \circ H_1^{c_0, c'_0}(-) & \xrightarrow{[\beta \bullet^1 \sigma]_1^{c_0, c'_0}} & F_1^{c_0, c'_0}(-) \circ \beta c'_0 \circ \sigma c'_0 \end{array}$$

$[\Lambda \bullet^1 \sigma]_{c_0 \circ H_1^{c_0, c'_0}} = \Lambda c_0 \circ \sigma c_0 \circ H_1^{c_0, c'_0}$

 $F_1^{c_0, c'_0} \circ [\Lambda \bullet^1 \sigma]_{c'_0} = F_1^{c_0, c'_0} \circ \Lambda c'_0 \circ \sigma c'_0$

The pair $\langle [\Lambda \bullet^1 \sigma]_0, [\Lambda \bullet^1 \sigma]_1^{-, -} \rangle$ forms indeed a 3-morphism of n -categories. The proof is a straightforward variation of the proof for reduced right-composition above, hence it is omitted.

5.5.3 Properties

In this section we give some properties of left/right 1-composition of a 3-morphism with a 2-morphism. They are modeled on similar properties given in the definition of a sesqui-category, and they are extremely useful in dealing with calculations. Let us consider the diagram

$$\begin{array}{ccccccc} & & & \alpha & & & \\ & & & \Downarrow \Lambda & & & \\ E' & \xRightarrow{\omega'} & E & \xRightarrow{\omega} & F & \xRightarrow{\beta} & G & \xRightarrow{\sigma} & H & \xRightarrow{\sigma'} & H' \\ & & & \Uparrow \Sigma & & & \\ & & & \gamma & & & \end{array}$$

as a reference for the following

Proposition 5.6. *(2-composition (i.e. vertical) composition of 3-morphisms w.r.t. (reduced) 1-composition with a 2-morphism)*

$$\begin{aligned}
(L1)' \quad id_F \bullet_L^1 \Lambda &= \Lambda & (R1)' \quad \Lambda \bullet_R^1 id_G &= \Lambda \\
(L2)' \quad (\omega' \bullet_L^1 \omega) \bullet_L^1 \Lambda &= \omega' \bullet_L^1 (\omega \bullet_L^1 \Lambda) & (R2)' \quad \Lambda \bullet_R^1 (\sigma \bullet^1 \sigma') &= (\Lambda \bullet_R^1 \sigma) \bullet_R^1 \sigma' \\
(L3)' \quad \omega \bullet_L^1 id_\alpha &= id_{\omega\alpha} & (R3)' \quad id_\alpha \bullet_R^1 \sigma &= id_{\alpha\sigma} \\
(L4)' \quad \omega \bullet_L^1 (\Lambda \bullet^2 \Sigma) &= (\omega \bullet_L^1 \Lambda) \bullet^2 (\omega \bullet_L^1 \Sigma) & (R4)' \quad (\Lambda \bullet^2 \Sigma) \bullet_R^1 \sigma &= (\Lambda \bullet_R^1 \sigma) \bullet^2 (\Sigma \bullet_R^1 \sigma) \\
(LR5)' \quad (\omega \bullet_L^1 \Lambda) \bullet_R^1 \sigma &= \omega \bullet_L^1 (\Lambda \bullet_R^1 \sigma)
\end{aligned}$$

Proof. We only prove statements $(L1)'$ to $(L4)'$ and statement $(LR5)$. Proof of properties $(R1)'$ to $(R4)'$ is just a straightforward variation of proof of $(L1)'$ to $(L4)'$, hence it will be omitted.

• $(L1)'$ (on objects) Let an object c_0 of \mathbb{C} be given. Then

$$[id_F \bullet^1 \Lambda]_{c_0} = [id_F]_{c_0} \circ \Lambda_{c_0} = id_{Fc_0} \circ \Lambda_{c_0} = \Lambda_{c_0}$$

(on homs) Let objects c_0, c'_0 of \mathbb{C} be given. Then

$$\begin{aligned}
[id_F \bullet^1 \Lambda]_1^{c_0, c'_0} &= ([id_F]_{c_0} \circ \Lambda_1^{c_0, c'_0}) \bullet^1 ([id_F]_1^{c_0, c'_0} \circ \beta_{c'_0}) \\
&= (id_{Fc_0} \circ \Lambda_1^{c_0, c'_0}) \bullet^1 (id_{F_1^{c_0, c'_0}} \circ \beta_{c'_0}) \\
&\stackrel{(\clubsuit)}{=} \Lambda_1^{c_0, c'_0} \bullet^1 id_{F_1^{c_0, c'_0} \circ \beta_{c'_0}} \\
&\stackrel{(\spadesuit)}{=} \Lambda_1^{c_0, c'_0}
\end{aligned}$$

where (\clubsuit) holds by $(R3)$ and (\spadesuit) holds by $(L1)'$ in dimension $n-1$.

• $(L2)'$ (on objects) Let an object c_0 of \mathbb{C} be given. Then

$$\begin{aligned}
[(\omega' \bullet^1 \omega) \bullet^1 \Lambda]_{c_0} &= [\omega' \bullet^1 \omega]_{c_0} \circ \Lambda_{c_0} \\
&= \omega'_{c_0} \circ \omega_{c_0} \circ \Lambda_{c_0} \\
&= \omega'_{c_0} \circ [\omega \bullet^1 \Lambda]_{c_0} \\
&= [\omega'_{c_0} \bullet^1 (\omega \bullet^1 \Lambda)]_{c_0}
\end{aligned}$$

(on homs) Let objects c_0, c'_0 of \mathbb{C} be given. Then

$$[(\omega' \bullet^1 \omega) \bullet^1 \Lambda]_1^{c_0, c'_0} =$$

$$\begin{aligned}
&= ([\omega' \bullet^1 \omega]_{c_0} \circ \Lambda_1^{c_0, c'_0}) \bullet_1 ([\omega' \bullet^1 \omega]_1^{c_0, c'_0} \circ \beta_{c'_0}) \\
&= (\omega'_{c_0} \circ \omega_{c_0} \circ \Lambda_1^{c_0, c'_0}) \bullet_1 \left(([\omega'_{c_0} \circ \omega_1^{c_0, c'_0}] \bullet^1 (\omega_1^{c_0, c'_0} \circ \omega_{c'_0})) \circ \beta_{c'_0} \right) \\
&= (\omega'_{c_0} \circ \omega_{c_0} \circ \Lambda_1^{c_0, c'_0}) \bullet_1 (\omega'_{c_0} \circ \omega_1^{c_0, c'_0} \circ \beta_{c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \omega_{c'_0} \circ \beta_{c'_0}) \quad (\spadesuit) \\
&= \left(\omega'_{c_0} \circ ([\omega_{c_0} \circ \Lambda_1^{c_0, c'_0}] \bullet^1 (\omega_1^{c_0, c'_0} \circ \beta_{c'_0})) \right) \bullet^1 (\omega_1^{c_0, c'_0} \circ (\omega_{c'_0} \circ \beta_{c'_0})) \\
&= (\omega'_{c_0} \circ [\omega \bullet^1 \Lambda]_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ [\omega \bullet^1 \beta]_{c'_0}) \\
&= [\omega' \bullet^1 (\omega \bullet^1 \Lambda)]_1^{c_0, c'_0}
\end{aligned}$$

where expression (\spadesuit) is unambiguous for the same property in dimension $n-1$.

• $(L3)'$ (on objects) Let an object c_0 of \mathbb{C} be given. Then quite plainly

$$[\omega \bullet^1 id_\alpha]_{c_0} = \omega_{c_0} \circ [id_\alpha]_{c_0} = \omega_{c_0} \circ id_{\alpha_{c_0}} = id_{\omega_{c_0} \circ \alpha_{c_0}} = id_{[\omega \bullet^1 \alpha]_{c_0}} = [id_{\omega \bullet^1 \alpha}]_{c_0}$$

(on homs) Let objects c_0, c'_0 of \mathbb{C} be given. Then

$$\begin{aligned}
[\omega \bullet^1 id_\alpha]_1^{c_0, c'_0} &= (\omega_{c_0} \circ [id_\alpha]_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \alpha_{c'_0}) \\
&= [id_{\omega_{c_0} \circ \alpha}]_1^{c_0, c'_0} \bullet^1 (\omega_1^{c_0, c'_0} \circ \alpha_{c'_0}) \\
&= (id_{\omega_{c_0} \circ \alpha_1^{c_0, c'_0}}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \alpha_{c'_0}) \\
&\stackrel{(\clubsuit)}{=} id_{(\omega_{c_0} \circ \alpha_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \alpha_{c'_0})} \\
&= id_{[\omega \bullet^1 \alpha]_1^{c_0, c'_0}} \\
&= [id_{\omega \bullet^1 \alpha}]_1^{c_0, c'_0}
\end{aligned}$$

where (\clubsuit) holds by $(L3)'$ in dimension $n-1$.

• $(L4)'$ (on objects) Let an object c_0 of \mathbb{C} be given. Then

$$\begin{aligned}
[\omega \bullet^1 (\Lambda \bullet^2 \Sigma)]_{c_0} &= \omega_{c_0} \circ^0 [\Lambda \bullet^2 \Sigma]_{c_0} \\
&= \omega_{c_0} \circ^0 (\Lambda_{c_0} \circ^1 \Sigma_{c_0}) \\
&\stackrel{(\heartsuit)}{=} (\omega_{c_0} \circ^0 \Lambda_{c_0}) \circ^1 (\omega_{c_0} \circ^0 \Sigma_{c_0}) \\
&= [\omega \bullet^1 \Lambda]_{c_0} \circ^1 [\omega \bullet^1 \Sigma]_{c_0} \\
&= [(\omega \bullet^1 \Lambda) \bullet^2 (\omega \bullet^1 \Sigma)]_{c_0}
\end{aligned}$$

where all equalities follow straightforward from definitions, but (\heartsuit) that is the *strict interchange property* of \circ^0 and \circ^1 .

(on homs) Let objects c_0, c'_0 of \mathbb{C} be given. Then by definition

$$[\omega \bullet^1 (\Lambda \bullet^2 \Sigma)]_1^{c_0, c'_0} = (\omega_{c_0} \circ [\Lambda \bullet^2 \Sigma]_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \gamma_{c'_0})$$

$$= \left(\omega_{c_0} \circ \left((\Lambda_{c_0} \circ G_1^{c_0, c'_0}) \bullet^1 \Sigma_1^{c_0, c'_0} \right) \bullet^2 (\Lambda_1^{c_0, c'_0} \bullet^1 (F_1^{c_0, c'_0} \circ \Sigma_{c'_0})) \right) \bullet^1 (\omega_1^{c_0, c'_0} \circ \gamma_{c'_0})$$

by 0-whiskering of a morphism with a 2-composition $(L4)''$ this gets

$$\left(\left(\omega_{c_0} \circ (\Lambda_{c_0} \circ G_1^{c_0, c'_0}) \bullet^1 \Sigma_1^{c_0, c'_0} \right) \bullet^2 \left(\omega_{c_0} \circ (\Lambda_1^{c_0, c'_0} \bullet^1 (F_1^{c_0, c'_0} \circ \Sigma_{c'_0})) \right) \right) \bullet^1 (\omega_1^{c_0, c'_0} \circ \gamma_{c'_0})$$

by 1-whiskering of a 2-morphism with a 2-composition $(R4)'$

$$\begin{aligned} & \left(\omega_{c_0} \circ (\Lambda_{c_0} \circ G_1^{c_0, c'_0}) \bullet^1 \Sigma_1^{c_0, c'_0} \right) \bullet^1 (\omega_1^{c_0, c'_0} \circ \gamma_{c'_0}) \\ & \quad \bullet^2 \\ & \left(\omega_{c_0} \circ (\Lambda_1^{c_0, c'_0} \bullet^1 (F_1^{c_0, c'_0} \circ \Sigma_{c'_0})) \right) \bullet^1 (\omega_1^{c_0, c'_0} \circ \gamma_{c'_0}) \end{aligned}$$

and by $(L4)$ and associativity of 1-composition (LR whiskering property)

$$\begin{aligned} & (\omega_{c_0} \circ \Lambda_{c_0} \circ G_1^{c_0, c'_0}) \bullet^1 (\omega_{c_0} \circ \Sigma_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \gamma_{c'_0}) \\ & \quad \bullet^2 \\ & (\omega_{c_0} \circ \Lambda_1^{c_0, c'_0}) \bullet^1 (\omega_{c_0} \circ F_1^{c_0, c'_0} \circ \Sigma_{c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \gamma_{c'_0}) \end{aligned}$$

rearranging the terms (since interchange holds for 0-composition with a constant transformation)

$$\begin{aligned} & (\omega_{c_0} \circ \Lambda_{c_0} \circ G_1^{c_0, c'_0}) \bullet^1 (\omega_{c_0} \circ \Sigma_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \gamma_{c'_0}) \\ & \quad \bullet^2 \\ & (\omega_{c_0} \circ \Lambda_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \Lambda_{c'_0}) \end{aligned}$$

rearranging the terms again

$$\begin{aligned} & (\omega_{c_0} \circ \Lambda_{c_0} \circ G_1^{c_0, c'_0}) \bullet^1 (\omega_{c_0} \circ \Sigma_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \gamma_{c'_0}) \\ & \quad \bullet^2 \\ & (\omega_{c_0} \circ \Lambda_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \beta_{c'_0}) \bullet^1 (E_1^{c_0, c'_0} \circ \omega_{c'_0} \circ \Sigma_{c'_0}) \end{aligned}$$

just adding brackets

$$\begin{aligned} & (\omega_{c_0} \circ \Lambda_{c_0} \circ G_1^{c_0, c'_0}) \bullet^1 \left((\omega_{c_0} \circ \Sigma_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \gamma_{c'_0}) \right) \\ & \quad \bullet^2 \\ & \left((\omega_{c_0} \circ \Lambda_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \beta_{c'_0}) \right) \bullet^1 (E_1^{c_0, c'_0} \circ \omega_{c'_0} \circ \Sigma_{c'_0}) \end{aligned}$$

by definition of 1-whiskering with a 3-morphism, this can be rewritten more conveniently

$$\left(([\omega \bullet^1 \Lambda]_{c_0} \circ G_1^{c_0, c'_0}) \bullet^1 [\omega \bullet^1 \Sigma]_1^{c_0, c'_0} \right) \bullet^2 \left([\omega \bullet^1 \Lambda]_1^{c_0, c'_0} \bullet^1 (E_1^{c_0, c'_0} \circ [\omega \bullet^1 \Sigma]_{c'_0}) \right)$$

and finally, by definition of 2-composition of 3-morphisms

$$[(\omega \bullet^1 \Lambda) \bullet^2 (\omega \bullet^1 \Sigma)]_1^{c_0, c'_0}$$

• $(LR5)'$ (*on objects*) Let an object c_0 of \mathbb{C} be given. Then

$$\begin{aligned} [(\omega \bullet^1 \Lambda) \bullet^1 \sigma]_{c_0} &= [\omega \bullet^1 \Lambda]_{c_0} \circ \sigma_{c_0} \\ &= \omega_{c_0} \circ \Lambda_{c_0} \circ \sigma_{c_0} \\ &= \omega_{c_0} \circ [\Lambda \bullet^1 \sigma]_{c_0} \\ &= [\omega \bullet^1 (\Lambda \bullet^1 \sigma)]_{c_0} \end{aligned}$$

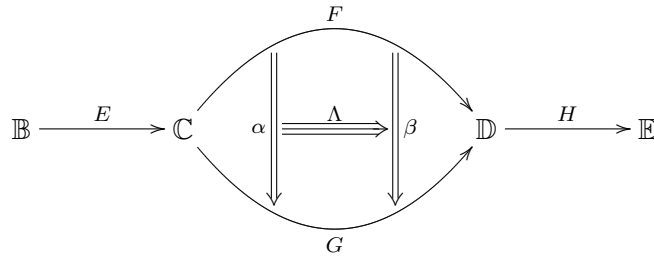
(*on homs*) Let objects c_0, c'_0 of \mathbb{C} be given. Then

$$\begin{aligned} [(\omega \bullet^1 \Lambda) \bullet^1 \sigma]_1^{c_0, c'_0} &= \\ &= ([\omega \bullet^1 \alpha]_{c_0} \circ \sigma_1^{c_0, c'_0}) \bullet^1 ([\omega \bullet^1 \Lambda]_1^{c_0, c'_0} \circ \sigma_{c'_0}) \\ &= (\omega_{c_0} \circ \alpha_{c_0} \circ \sigma_1^{c_0, c'_0}) \bullet^1 \left(((\omega_{c_0} \circ \Lambda_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \beta_{c'_0})) \circ \sigma_{c'_0} \right) \\ &= (\omega_{c_0} \circ \alpha_{c_0} \circ \sigma_1^{c_0, c'_0}) \bullet^1 (\omega_{c_0} \circ \Lambda_1^{c_0, c'_0} \circ \sigma_{c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ \beta_{c'_0} \circ \sigma_{c'_0}) \\ &= \left(\omega_{c_0} \circ ((\alpha_{c_0} \circ \sigma_1^{c_0, c'_0}) \bullet^1 (\Lambda_1^{c_0, c'_0} \circ \sigma_{c'_0})) \right) \bullet^1 (\omega_1^{c_0, c'_0} \circ \beta_{c'_0} \circ \sigma_{c'_0}) \\ &= (\omega_{c_0} \circ [\Lambda \bullet^1 \Sigma]_1^{c_0, c'_0}) \bullet^1 (\omega_1^{c_0, c'_0} \circ [\beta \bullet^1 \sigma]_{c'_0}) \\ &= [\omega \bullet^1 (\Lambda \bullet^1 \sigma)]_1^{c_0, c'_0} \end{aligned}$$

□

5.6 0-whiskering of 3-morphisms

In this section we define reduced left/right 1-composition of a 3-morphism with a 1-morphism, according to the following reference diagram.



5.6.1 Reduced left-composition

The 3-morphism

$$E \bullet^0 \Lambda : E \bullet^0 \alpha \Rightarrow E \bullet^0 \beta : E \bullet^0 F \Rightarrow E \bullet^0 G : \mathbb{B} \rightarrow \mathbb{D}$$

is given by the data below

- (on objects) For an object b_0 of \mathbb{B}

$$[E \bullet^0 \Lambda]_0 : b_0 \mapsto \alpha(Eb_0) \begin{array}{c} \xrightarrow{F(Eb_0)} \\ \Lambda(Eb_0) \\ \xrightarrow{G(Eb_0)} \end{array} \beta(Eb_0)$$

- (on homs) For objects b_0, b'_0 of \mathbb{B}

$$[E \bullet^0 \Lambda]_1^{b_0, b'_0} = E_1^{b_0, b'_0}(-) \bullet^0 \Lambda_1^{Eb_0, Eb'_0}$$

$$\begin{array}{ccc} \alpha(Eb_0) \circ G_1^{Eb_0, Eb'_0} \left(E_1^{b_0, b'_0}(-) \right) & \xRightarrow{\Lambda(Eb_0) \circ id} & \beta(Eb_0) \circ G_1^{Eb_0, Eb'_0} \left(E_1^{b_0, b'_0}(-) \right) \\ \downarrow [E \bullet^0 \alpha]_1^{b_0, b'_0} & \swarrow [E \bullet^0 \Lambda]_1^{b_0, b'_0} & \downarrow [E \bullet^0 \beta]_1^{b_0, b'_0} \\ = E_1^{b_0, b'_0} \bullet^0 \alpha_1^{Eb_0, Eb'_0} & & = E_1^{b_0, b'_0} \bullet^0 \beta_1^{Eb_0, Eb'_0} \\ \downarrow & & \downarrow \\ F_1^{Eb_0, Eb'_0} \left(E_1^{b_0, b'_0}(-) \right) \circ \alpha(Eb'_0) & \xRightarrow{id \circ \Lambda(Eb'_0)} & F_1^{Eb_0, Eb'_0} \left(E_1^{b_0, b'_0}(-) \right) \circ \beta(Eb'_0) \end{array}$$

The pair $\langle [E \bullet^0 \alpha]_0, [E \bullet^0 \alpha]_1^{-, -} \rangle$ forms indeed a 3-morphism of n-categories.

Proof. We have to show that it satisfies composition and unit axioms. Let us begin with composition, and fix a triple c_0, c'_0, c''_0 of \mathbb{C} . Notice that, in order to keep wide diagrams in the page we write \circ^0 -composition by juxtaposition, and subscripts for transformations on objects are used.

$$\begin{array}{ccc}
\alpha_{Eb_0}[EG]_1^{b_0,b'_0}[EG]_1^{b'_0,b''_0} & \xrightarrow{\Lambda_{Eb_0} id id} & \beta_{Eb_0}[EG]_1^{b_0,b'_0}[EG]_1^{b'_0,b''_0} \\
\downarrow [E \bullet^0 \alpha]_1^{b_0,b'_0} id & \swarrow [E \bullet^0 \Lambda]_1^{b_0,b'_0} id & \downarrow [E \bullet^0 \beta]_1^{b_0,b'_0} id \\
[EF]_1^{b_0,b'_0} \alpha_{Eb'_0}[EG]_1^{b'_0,b''_0} & \xrightarrow{id \Lambda_{Eb'_0} id} & [EF]_1^{Eb_0,Eb'_0} \beta_{Eb'_0}[EG]_1^{b'_0,b''_0} \\
\downarrow id [E \bullet^0 \alpha]_1^{b'_0,b''_0} & \swarrow id [E \bullet^0 \Lambda]_1^{b'_0,b''_0} & \downarrow id [E \bullet^0 \beta]_1^{b'_0,b''_0} \\
[EF]_1^{b_0,b'_0}[EG]_1^{b'_0,b''_0} \alpha_{Eb''_0} & \xrightarrow{id id \Lambda_{Eb''_0}} & [EF]_1^{Eb_0,Eb'_0}[EG]_1^{b'_0,b''_0} \beta_{Eb''_0}
\end{array}$$

This can be written equationally

$$\begin{aligned}
& \left[\left(E_1^{b_0,b'_0} \bullet^0 \Lambda_1^{Eb_0,Eb'_0} \right) \circ \left(E_1^{b'_0,b''_0} \bullet^0 G_1^{Eb'_0,Eb''_0} \right) \right] \bullet^1 \left[\left(E_1^{b_0,b'_0} \bullet^0 F_1^{Eb_0,Eb'_0} \right) \circ \left(E_1^{b'_0,b''_0} \bullet^0 \beta_1^{Eb'_0,Eb''_0} \right) \right] \\
& \quad \bullet^2 \\
& \left[\left(E_1^{b_0,b'_0} \bullet^0 \alpha_1^{Eb_0,Eb'_0} \right) \circ \left(E_1^{b'_0,b''_0} \bullet^0 G_1^{Eb'_0,Eb''_0} \right) \right] \bullet^1 \left[\left(E_1^{b_0,b'_0} \bullet^0 F_1^{Eb_0,Eb'_0} \right) \circ \left(E_1^{b'_0,b''_0} \bullet^0 \Lambda_1^{Eb'_0,Eb''_0} \right) \right]
\end{aligned}$$

by product interchange before \circ^0 -composition this turns to be

$$\begin{aligned}
& \left[\left(E_1^{b_0,b'_0} \times E_1^{b'_0,b''_0} \right) \bullet^0 \left(\Lambda_1^{Eb_0,Eb'_0} \circ G_1^{Eb'_0,Eb''_0} \right) \right] \bullet^1 \left[\left(E_1^{b_0,b'_0} \times E_1^{b'_0,b''_0} \right) \bullet^0 \left(F_1^{Eb_0,Eb'_0} \circ \beta_1^{Eb'_0,Eb''_0} \right) \right] \\
& \quad \bullet^2 \\
& \left[\left(E_1^{b_0,b'_0} \times E_1^{b'_0,b''_0} \right) \bullet^0 \left(\alpha_1^{Eb_0,Eb'_0} \circ G_1^{Eb'_0,Eb''_0} \right) \right] \bullet^1 \left[\left(E_1^{b_0,b'_0} \times E_1^{b'_0,b''_0} \right) \bullet^0 \left(F_1^{Eb_0,Eb'_0} \circ \Lambda_1^{Eb'_0,Eb''_0} \right) \right]
\end{aligned}$$

by *whiskering interchange property* (LR) this gives

$$\begin{aligned}
& \left(E_1^{b_0,b'_0} \times E_1^{b'_0,b''_0} \right) \bullet^0 \left[\left(\Lambda_1^{Eb_0,Eb'_0} \circ G_1^{Eb'_0,Eb''_0} \right) \bullet^1 \left(F_1^{Eb_0,Eb'_0} \circ \beta_1^{Eb'_0,Eb''_0} \right) \right] \\
& \quad \bullet^2 \\
& \left(E_1^{b_0,b'_0} \times E_1^{b'_0,b''_0} \right) \bullet^0 \left[\left(\alpha_1^{Eb_0,Eb'_0} \circ G_1^{Eb'_0,Eb''_0} \right) \bullet^1 \left(F_1^{Eb_0,Eb'_0} \circ \Lambda_1^{Eb'_0,Eb''_0} \right) \right]
\end{aligned}$$

hence by (L4)''

$$(E_1^{b_0, b'_0} \times E_1^{b'_0, b''_0}) \bullet^0 \left(\begin{array}{c} \left(\Lambda_1^{Eb_0, Eb'_0} \circ G_1^{Eb'_0, Eb''_0} \right) \bullet^1 \left(F_1^{Eb_0, Eb'_0} \circ \beta_1^{Eb'_0, Eb''_0} \right) \\ \bullet^2 \\ \left(\alpha_1^{Eb_0, Eb'_0} \circ G_1^{Eb'_0, Eb''_0} \right) \bullet^1 \left(F_1^{Eb_0, Eb'_0} \circ \Lambda_1^{Eb'_0, Eb''_0} \right) \end{array} \right)$$

By composition functoriality of Λ , the second row changes to $(- \circ -) \bullet^0 \Lambda_1^{Eb_0, Eb''_0}$ thus giving by associativity of 0-whiskering

$$(E_1^{b_0, b'_0} \circ E_1^{b'_0, b''_0}) \bullet^0 \Lambda_1^{Eb_0, Eb''_0} = E_1^{b_0, b''_0} \bullet^0 \Lambda_1^{Eb_0, Eb''_0}$$

where the last equality follows from composition axiom for 1-morphisms.

Turning to units axiom, let an object b_0 of \mathbb{B} be given. Then

$$u(b_0) \bullet^0 E_1^{b_0, b_0} \bullet^0 \Lambda_1^{Eb_0, Eb_0} = u(Eb_0) \bullet^0 \Lambda_1^{Eb_0, Eb_0} = id_{[\Lambda(Eb_0)]} = id_{[(E \bullet^0 \Lambda)(b_0)]}$$

where the first expression is unambiguous for \bullet^0 -associativity $(L2)''$, first equality holds by units axiom for 1-morphism E , second by units axiom for 3-morphism Λ , last is the definition. \square

5.6.2 Reduced right-composition

The 3-morphism

$$\Lambda \bullet^0 H : \alpha \bullet^0 \Rightarrow \beta \bullet^0 H : F \bullet^0 H \Rightarrow G \bullet^0 H : \mathbb{C} \rightarrow \mathbb{E}$$

is given by the data below

- (on objects) For an object c_0 of \mathbb{C}

$$[\Lambda \bullet^0 H]_0 : c_0 \mapsto H(\alpha c_0) \begin{array}{c} \xrightarrow{H(Fc_0)} \\ \xRightarrow{H(\Lambda c_0)} \\ \xrightarrow{H(Gc_0)} \end{array} H(\beta c_0)$$

- (on homs) For objects c_0, c'_0 of \mathbb{C}

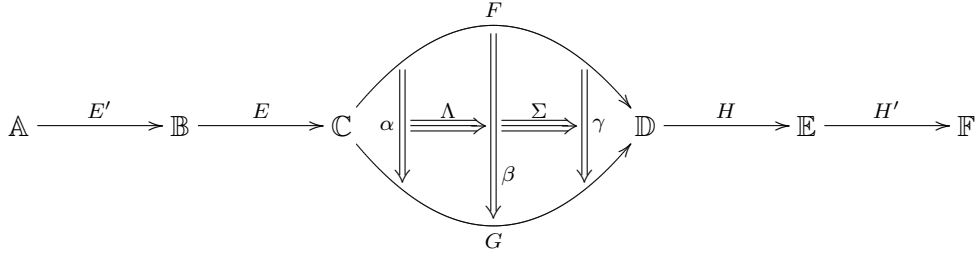
$$[\Lambda \bullet^0 H]_1^{c_0, c'_0} = \Lambda_1^{c_0, c'_0} \bullet^0 H_1^{Fc_0, Gc'_0}$$

$$\begin{array}{ccc}
H_1^{F_{c_0}, G_{c_0}}(\alpha_{c_0}) \circ H_1^{G_{c_0}, G_{c'_0}}(G_1^{c_0, c'_0}(-)) & \xrightarrow{H_1^{F_{c_0}, G_{c_0}}(\Lambda_{c_0}) \circ id} & H_1^{F_{c_0}, G_{c_0}}(\beta_{c_0}) \circ H_1^{G_{c_0}, G_{c'_0}}(G_1^{c_0, c'_0}(-)) \\
\downarrow H_1^{F_{c_0}, G_{c'_0}}(\alpha_1^{c_0, c'_0}) & \swarrow [\Lambda \bullet^0 H]_1^{c_0, c'_0} & \downarrow H_1^{F_{c_0}, G_{c'_0}}(\beta_1^{c_0, c'_0}) \\
H_1^{F_{c_0}, F_{c'_0}}(F_1^{c_0, c'_0}) \circ H_1^{F_{c'_0}, G_{c'_0}}(\alpha_{c'_0}) & \xrightarrow{id \circ H_1^{F_{c'_0}, G_{c'_0}}(\Lambda_{c'_0})} & H_1^{F_{c_0}, F_{c'_0}}(F_1^{c_0, c'_0}) \circ H_1^{F_{c'_0}, G_{c'_0}}(\beta_{c'_0})
\end{array}$$

The pair $\langle [\Lambda \bullet^0 H]_0, [\Lambda \bullet^0 H]_1^{-, -} \rangle$ forms indeed a 3-morphism of n-categories. The proof is a straightforward variation of the proof for reduced right-composition above, hence it is omitted.

5.6.3 Properties

As we did in describing the sesqui-categorical structure for homs in $n\mathbf{Cat}$, we use again a *left-and-right* approach to describe properties of the 0-whiskering of a 3-morphism with a morphism. Let us consider the diagram



as a reference for the following

Proposition 5.7 (2-composition (i.e. vertical) composition of 3-morphisms w.r.t. (reduced) 0-composition with a (1-)morphism).

$$\begin{array}{ll}
(L1)'' & id_{\mathbb{C}} \bullet_L^0 \Lambda = \Lambda \\
(L2)'' & (E' \bullet_L^0 E) \bullet_L^0 \Lambda = E' \bullet_L^0 (E \bullet_L^0 \Lambda) \\
(L3)'' & E \bullet_L^0 id_{\alpha} = id_{E \bullet_L^0 \alpha} \\
(L4)'' & E \bullet_L^0 (\Lambda \bullet^2 \Sigma) = (E \bullet_L^0 \Lambda) \bullet^2 (E \bullet_L^0 \Sigma) \\
(R1)'' & \Lambda \bullet_R^0 id_{\mathbb{D}} = \Lambda \\
(R2)'' & \Lambda \bullet_R^0 (H \bullet_R^0 H') = (\Lambda \bullet_R^0 H) \bullet_R^0 H' \\
(R3)'' & id_{\alpha} \bullet_R^0 H = id_{\alpha \bullet_R^0 H} \\
(R4)'' & (\Lambda \bullet^2 \Sigma) \bullet_R^0 H = (\Lambda \bullet_R^0 H) \bullet^2 (\Sigma \bullet_R^0 H) \\
(LR5)'' & (E \bullet_L^0 \Lambda) \bullet_R^0 H = E \bullet_L^0 (\Lambda \bullet_R^0 H)
\end{array}$$

Proof. We prove statements (L1)'' to (L4)'' and (LR5). Proofs of statements (R1)'' to (R4)'' is similar, hence it is omitted.

- (L1)'' (on objects) Let an object c_0 of \mathbb{C} be given. Then

$$[id_{\mathbb{C}} \bullet^0 \Lambda]_{c_0} = \Lambda(id_{\mathbb{C}}(c_0)) = \Lambda_{c_0}$$

(on homs) Let objects c_0, c'_0 of \mathbb{C} be given. Then

$$[id_{\mathbb{C}} \bullet^0 \Lambda]_1^{c_0 c'_0} = [id_{\mathbb{C}}]_1^{c_0, c'_0} \bullet^0 \Lambda_1^{c_0 c'_0} = id_{\mathbb{C}_1(c_0, c'_0)} \bullet^0 \Lambda_1^{c_0 c'_0} = \Lambda_1^{c_0 c'_0}$$

- (L2)'' (on objects) Let an object a_0 of \mathbb{A} be given. Then

$$[(E' \bullet^0 E) \bullet^0 \Lambda]_{a_0} = \Lambda_{(E' \bullet^0 E)_{a_0}} = \Lambda_{E(E' a_0)} = [E \bullet^0 \Lambda]_{E' a_0} = [E' \bullet^0 (E \bullet^0 \Lambda)]_{a_0}$$

(on homs) Let objects a_0, a'_0 of \mathbb{A} be given. Then

$$\begin{aligned} [(E' \bullet^0 E) \bullet^0 \Lambda]_1^{a_0, a'_0} &= [E' \bullet^0 E]_1^{a_0, a'_0} \bullet^0 \Lambda_1^{E(E' a_0), E(E' a'_0)} \\ &= E_1^{a_0, a'_0} \bullet^0 E_1^{E' a_0, E' a'_0} \bullet^0 \Lambda_1^{E(E' a_0), E(E' a'_0)} \\ &= E_1^{a_0, a'_0} \bullet^0 [E \bullet^0 \Lambda]_1^{E' a_0, E' a'_0} \\ &= [E' \bullet^0 (E \bullet^0 \Lambda)]_1^{a_0, a'_0} \end{aligned}$$

- (L3)'' (on objects) Let an object b_0 of \mathbb{B} be given. Then

$$[E \bullet^0 id_{\alpha}]_{b_0} = [id_{\alpha}]_{Eb_0} = id_{\alpha_{Eb_0}} = id_{[E \bullet^0 \alpha]_{b_0}} = [id_{E \bullet^0 \alpha}]_{b_0}$$

(on homs) Let objects b_0, b'_0 of \mathbb{B} be given. Then

$$\begin{aligned} [E \bullet^0 id_{\alpha}]_1^{b_0, b'_0} &= E_1^{b_0, b'_0} \bullet^0 [id_{\alpha}]_1^{Eb_0, Eb'_0} \\ &= E_1^{b_0, b'_0} \bullet^0 id_{\alpha_1^{Eb_0, Eb'_0}} \\ &\stackrel{(\clubsuit)}{=} id_{E_1^{b_0, b'_0} \bullet^0 \alpha_1^{Eb_0, Eb'_0}} \\ &= id_{[E \bullet^0 \alpha]_1^{b_0, b'_0}} \\ &= [id_{E \bullet^0 \alpha}]_1^{b_0, b'_0} \end{aligned}$$

where (\clubsuit) holds for the same property in dimension $n-1$.

- (L4)'' (on objects) Let an object b_0 of \mathbb{B} be given. Then

$$\begin{aligned} [E \bullet^0 (\Lambda \bullet^2 \Sigma)]_{b_0} &= [(\Lambda \bullet^2 \Sigma)]_{Eb_0} \\ &= \Lambda_{Eb_0} \circ^1 \Sigma_{Eb_0} \\ &= [E \bullet^0 \Lambda]_{b_0} \circ^1 [E \bullet^0 \Sigma]_{b_0} \\ &= [(E \bullet^0 \Lambda) \bullet^2 (E \bullet^0 \Sigma)]_{b_0} \end{aligned}$$

(on homs) Let objects b_0, b'_0 of \mathbb{B} be given. Then

$$\begin{aligned}
[E \bullet^0 (\Lambda \bullet^2 \Sigma)]_1^{b_0, b'_0} &= E_1^{b_0, b'_0} \bullet^0 [\Lambda \bullet^2 \Sigma]_1^{Eb_0, Eb'_0} \\
&= E_1^{b_0, b'_0} \bullet^0 \left(\begin{array}{c} (\Lambda_{Eb_0} \circ G_1^{Eb_0, Eb'_0}) \bullet^1 \Sigma_1^{Eb_0, Eb'_0} \\ \bullet^2 \\ \Lambda_1^{Eb_0, Eb'_0} \bullet^1 (F_1^{Eb_0, Eb'_0} \circ \Sigma_{Eb'_0}) \end{array} \right) \\
&\stackrel{(\heartsuit)}{=} \left(\begin{array}{c} E_1^{b_0, b'_0} \bullet^0 ((\Lambda_{Eb_0} \circ G_1^{Eb_0, Eb'_0}) \bullet^1 \Sigma_1^{Eb_0, Eb'_0}) \\ \bullet^2 \\ E_1^{b_0, b'_0} \bullet^0 (\Lambda_1^{Eb_0, Eb'_0} \bullet^1 (F_1^{Eb_0, Eb'_0} \circ \Sigma_{Eb'_0})) \end{array} \right) \\
&\stackrel{(\clubsuit)}{=} \left(\begin{array}{c} (\Lambda_{Eb_0} \circ (E_1^{b_0, b'_0} \bullet^0 G_1^{Eb_0, Eb'_0})) \bullet^1 (E_1^{b_0, b'_0} \bullet^0 \Sigma_1^{Eb_0, Eb'_0}) \\ \bullet^2 \\ (E_1^{b_0, b'_0} \bullet^0 \Lambda_1^{Eb_0, Eb'_0}) \bullet^1 ((E_1^{b_0, b'_0} \bullet^0 F_1^{Eb_0, Eb'_0}) \circ \Sigma_{Eb'_0}) \end{array} \right) \\
&= \left(\begin{array}{c} ([E \bullet^0 \Lambda]_{b_0} \circ [E \bullet^0 G]_1^{b_0, b'_0}) \bullet^1 [E \bullet^0 \Sigma]_1^{b_0, b'_0} \\ \bullet^2 \\ [E \bullet^0 \Lambda]_1^{b_0, b'_0} \bullet^1 ([E \bullet^0 F]_1^{b_0, b'_0} \circ [E \bullet^0 \Sigma]_{b'_0}) \end{array} \right) \\
&= [(E \bullet^0 \Lambda) \bullet^2 (E \bullet^0 \Sigma)]_1^{b_0, b'_0}
\end{aligned}$$

where (\heartsuit) holds by the same property in dimension $n - 1$, and (\clubsuit) holds by whiskering interchange property.

• $(LR5)''$ (on objects) Let an object b_0 of \mathbb{B} be given. Then

$$\begin{aligned}
[(E \bullet^0 \Lambda) \bullet^0 H]_{b_0} &= H([E \bullet^0 \Lambda]_{b_0}) = \\
&= H(\Lambda_{Eb_0}) \\
&= [\Lambda \bullet^0 H]_{Eb_0} \\
&= [E \bullet^0 (\Lambda \bullet^0 H)]_{b_0}
\end{aligned}$$

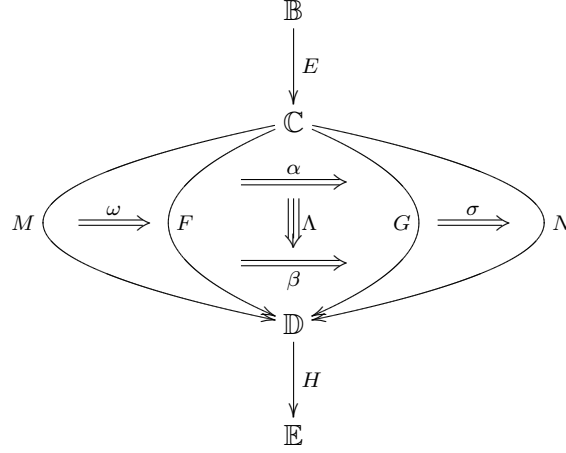
(on homs) Let objects b_0, b'_0 of \mathbb{B} be given. Then

$$\begin{aligned}
[(E \bullet^0 \Lambda) \bullet^0 H]_1^{b_0, b'_0} &= [E \bullet^0 \Lambda]_1^{b_0, b'_0} \bullet^0 H_1^{F(Eb_0), G(Eb'_0)} \\
&= E_1^{b_0, b'_0} \bullet^0 \Lambda_1^{Eb_0, Eb'_0} \bullet^0 H_1^{F(Eb_0), G(Eb'_0)} \\
&= E_1^{b_0, b'_0} \bullet^0 [\Lambda \bullet^0 H]_1^{Eb_0, Eb'_0} \\
&= [E \bullet^0 (\Lambda \bullet^0 H)]_1^{b_0, b'_0}
\end{aligned}$$

□

Before switching to next section, let us give a last property that express at once functoriality of left and right 0-composition with a morphism. To this end, let us be given also 2-morphisms $\omega : M \Rightarrow F$ and $\sigma : G \Rightarrow N$, as

represented in the diagram below



Left/right 0-composition of a 3-morphism with a morphism satisfies also the following property that relates 0-whiskering w.r.t. 1-whiskering:

Proposition 5.8 (Whiskering interchange property).

$$(LRW) \quad E \bullet_L^0 (\omega \bullet_L^1 \Lambda \bullet_R^1 \sigma) \bullet_R^0 H = (E \bullet_L^0 \omega \bullet_R^0 H) \bullet_L^1 (E \bullet_L^0 \Lambda \bullet_R^0 H) \bullet_R^1 (E \bullet_L^0 \sigma \bullet_R^0 H)$$

Proof. Without loss of generality it suffices to prove the following two equalities:

$$\begin{aligned} (LRW)^1 \quad E \bullet_L^0 (\omega \bullet_L^1 \Lambda) &= (E \bullet_L^0 \omega) \bullet_L^1 (E \bullet_L^0 \Lambda) \\ (LRW)^2 \quad E \bullet_L^0 (\Lambda \bullet_R^1 \sigma) &= (E \bullet_L^0 \Lambda) \bullet_R^1 (E \bullet_L^0 \sigma) \end{aligned}$$

• $(LRW)^1$ (on objects) Let an object b_0 of \mathbb{B} be given. Then

$$\begin{aligned} [E \bullet^0 (\omega \bullet^1 \Lambda)]_{b_0} &= [\omega \bullet^1 \Lambda]_{Eb_0} \\ &= \omega_{Eb_0} \circ \Lambda_{Eb_0} \\ &= [E \bullet^0 \omega]_{b_0} \circ [E \bullet^0 \Lambda]_{b_0} \\ &= [(E \bullet^0 \omega) \bullet^1 (E \bullet^0 \Lambda)]_{b_0} \end{aligned}$$

(on homs) Let objects b_0, b'_0 of \mathbb{B} be given. Then

$$[E \bullet^0 (\omega \bullet^1 \Lambda)]_1^{b_0, b'_0} =$$

$$\begin{aligned}
&= E_1^{b_0, b'_0} \bullet^0 [\omega \bullet^1 \Lambda]_1^{Eb_0, Eb'_0} \\
&= E_1^{b_0, b'_0} \bullet^0 \left((\omega_{Eb_0} \circ \Lambda_1^{Eb_0, Eb'_0}) \bullet^1 (\omega_1^{Eb_0, Eb'_0} \circ \beta_{Eb'_0}) \right) \\
(\clubsuit) \quad &= \left(E_1^{b_0, b'_0} \bullet^0 (\omega_{Eb_0} \circ \Lambda_1^{Eb_0, Eb'_0}) \right) \bullet^1 \left(E_1^{b_0, b'_0} \bullet^0 (\omega_1^{Eb_0, Eb'_0} \circ \beta_{Eb'_0}) \right) \\
&= \left(E_1^{b_0, b'_0} \bullet^0 \Lambda_1^{Eb_0, Eb'_0} \bullet^0 (\omega_{Eb_0} \circ -) \right) \bullet^1 \left(E_1^{b_0, b'_0} \bullet^0 \omega_1^{Eb_0, Eb'_0} \bullet^0 (- \circ \beta_{Eb'_0}) \right) \\
&= \left(\omega_{Eb_0} \circ (E_1^{b_0, b'_0} \bullet^0 \Lambda_1^{Eb_0, Eb'_0}) \right) \bullet^1 \left((E_1^{b_0, b'_0} \bullet^0 \omega_1^{Eb_0, Eb'_0}) \circ \beta_{Eb'_0} \right) \\
&= \left([E \bullet^0 \omega]_{b_0} \circ [E \bullet^0 \Lambda]_1^{b_0, b'_0} \right) \bullet^1 \left([E \bullet^0 \omega]_1^{b_0, b'_0} \circ [E \bullet^0 \beta]_{b'_0} \right) \\
&= [(E \bullet^0 \omega) \bullet^1 (E \bullet^0 \Lambda)]_1^{b_0, b'_0}
\end{aligned}$$

where equality (\clubsuit) holds by same property in dimension $n - 1$, and the following by associativity of 0-composition.

• $(LRW)^2$ (on objects) Let an object b_0 of \mathbb{B} be given. Then

$$\begin{aligned}
[E \bullet^0 (\Lambda \bullet^1 \sigma)]_{b_0} &= [\Lambda \bullet^1 \sigma]_{Eb_0} \\
&= \Lambda_{Eb_0} \circ \sigma_{Eb_0} \\
&= [E \bullet^0 \Lambda]_{b_0} \circ [E \bullet^0 \sigma]_{b_0} \\
&= [(E \bullet^0 \Lambda) \bullet^1 (E \bullet^0 \sigma)]_{b_0}
\end{aligned}$$

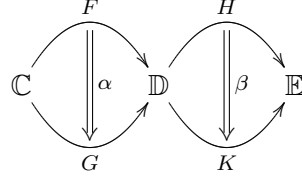
(on homs) Let objects b_0, b'_0 of \mathbb{B} be given. Then

$$\begin{aligned}
&[E \bullet^0 (\Lambda \bullet^1 \sigma)]_1^{b_0, b'_0} \\
&= E_1^{b_0, b'_0} \bullet^0 [\Lambda \bullet^1 \sigma]_1^{Eb_0, Eb'_0} \\
&= E_1^{b_0, b'_0} \bullet^0 \left((\alpha_{Eb_0} \circ \sigma_1^{Eb_0, Eb'_0}) \bullet^1 (\Lambda_1^{Eb_0, Eb'_0} \circ \sigma_{Eb'_0}) \right) \\
&= \left(E_1^{b_0, b'_0} \bullet^0 \sigma_1^{Eb_0, Eb'_0} (\alpha_{Eb_0} \circ -) \right) \bullet^1 \left(E_1^{b_0, b'_0} \bullet^0 \Lambda_1^{Eb_0, Eb'_0} (- \circ \sigma_{Eb'_0}) \right) \\
&= \left(E_1^{b_0, b'_0} \bullet^0 (\alpha_{Eb_0} \circ \sigma_1^{Eb_0, Eb'_0}) \right) \bullet^1 \left(E_1^{b_0, b'_0} \bullet^0 (\Lambda_1^{Eb_0, Eb'_0} \circ \sigma_{Eb'_0}) \right) \\
&= \left(\alpha_{Eb_0} \circ (E_1^{b_0, b'_0} \bullet^0 \sigma_1^{Eb_0, Eb'_0}) \right) \bullet^1 \left((E_1^{b_0, b'_0} \bullet^0 \Lambda_1^{Eb_0, Eb'_0}) \circ \sigma_{Eb'_0} \right) \\
&= \left([E \bullet^0 \alpha]_{b_0} \circ [E \bullet^0 \sigma]_1^{b_0, b'_0} \right) \bullet^1 \left([E \bullet^0 \Lambda]_1^{b_0, b'_0} \circ [E \bullet^0 \sigma]_{b'_0} \right) \\
&= [(E \bullet^0 \Lambda) \bullet^1 (E \bullet^0 \sigma)]_1^{b_0, b'_0}
\end{aligned}$$

□

5.7 Dimension raising 0-composition of 2-morphisms

Let two 0-intersecting 2-morphisms of n -categories be given.



It is easy to verify that in general

$$\alpha \setminus \beta := (F \bullet^0 \beta) \bullet^1 (\alpha \bullet^0 K) \neq (\alpha \bullet^0 H) \bullet^1 (G \bullet^1 \beta) =: \alpha / \beta$$

(5.5)

More interestingly they constitute a 3-morphism

$$\alpha * \beta : \alpha \setminus \beta \Longrightarrow \alpha / \beta$$

In fact, for every object c_0 of \mathbb{C} one defines

$$[\alpha * \beta]_0 : c_0 \mapsto \begin{array}{ccccc} & & H(Fc_0) & & \\ \beta Fc_0 \swarrow & & & \searrow H\alpha c_0 & \\ K(Fc_0) & \xrightarrow{\beta_1(\alpha c_0)} & H(Gc_0) & & \\ K\alpha c_0 \searrow & & & \swarrow \beta Gc_0 & \\ & & K(Gc_0) & & \end{array}$$

Moreover for every pair of objects c_0, c'_0 of \mathbb{C} one defines

$$[\alpha * \beta]_1^{c_0, c'_0} = \alpha_1^{c_0, c'_0} * \beta_1^{Fc_0, Gc'_0}$$

Claim 5.9. *The pair*

$$< [\alpha * \beta]_0, [\alpha * \beta]_1^{-, -} >$$

is indeed a 3-morphism of n -categories.

Proof. We must show that domain and codomain of $[\alpha * \beta]_1^{-,-}$ are compatible with those of the definition of 3-morphism. Moreover the pair above must satisfy unit and composition axioms. In order to prove the first fact, we write the diagram that represents the 3-morphism of (n-1)categories

$$[\alpha * \beta]_1^{c_0, c'_0} : \alpha_1^{c_0, c'_0} \setminus \beta_1^{F c_0, G c'_0} \Longrightarrow \alpha_1^{c_0, c'_0} / \beta_1^{F c_0, G c'_0}$$

i.e. the composition

$$\begin{array}{ccccc}
 & & [c_0, c'_0] & & \\
 & \swarrow F_1 & & \searrow G_1 & \\
 [F c_0, F c'_0] & & \xleftarrow{\alpha_1^{c_0, c'_0}} & & [G c_0, G c'_0] \\
 & \searrow -\circ \alpha c'_0 & & \swarrow \alpha c_0 \circ - & \\
 & & [F c_0, G c'_0] & & \\
 & \swarrow H_1 & & \searrow K_1 & \\
 [H(F c_0), H(G c'_0)] & & \xleftarrow{\beta_1^{F c_0, G c'_0}} & & [K(F c_0), K(G c'_0)] \\
 & \searrow -\circ \beta G c'_0 & & \swarrow \beta F c_0 \circ - & \\
 & & [H(F c_0), K(G c'_0)] & &
 \end{array}$$

Its domain is computed below

$$\begin{array}{ccccc}
 & & [c_0, c'_0] & & \\
 & \swarrow [FH]_1 & \downarrow [GH]_1 & \searrow \alpha c_0 \circ G_1(-) & \\
 [H(F c_0), H(F c'_0)] & & [H(G c_0), H(G c'_0)] & & [F c_0, G c'_0] \\
 \downarrow -\circ H \alpha c'_0 & \swarrow H \alpha c_0 \circ - & & \swarrow H_1 & \downarrow H_1 \\
 [H(F c_0), H(G c'_0)] & & [H(F c_0), H(G c'_0)] & \xleftarrow{\beta_1^{F c_0, G c'_0}} & [K(F c_0), K(G c'_0)] \\
 & \searrow -\circ \beta G c'_0 & \downarrow -\circ \beta G c'_0 & \swarrow \beta F c_0 \circ - & \\
 & & [H(F c_0), K(G c'_0)] & &
 \end{array}$$

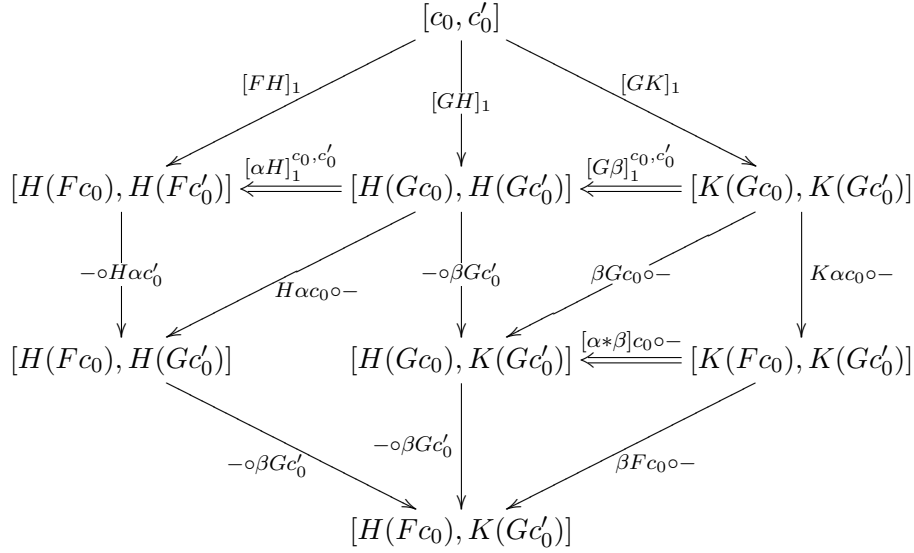
Now, by functoriality w.r.t 0-composition, with constant left composite one has

$$(\alpha c_0 \circ -) \bullet^0 \beta_1^{F c_0, G c'_0} = \left(K_1^{G c_0, G c'_0} \bullet^0 (\beta_1(\alpha c_0) \circ -) \right) \bullet^1 \left(\beta_1^{G c_0, G c'_0} \bullet^0 (H \alpha c_0 \circ -) \right)$$

and by definition of $*$ -composition on objects,

$$= \left(K_1^{Gc_0, Gc'_0} \bullet^0 ([\alpha * \beta]_{c_0} \circ -) \right) \bullet^1 \left(\beta_1^{Gc_0, Gc'_0} \bullet^0 (H\alpha c_0 \circ -) \right)$$

Hence we can redraw the domain



And this completes the domain-part. Concerning the codomain, the calculation is similar, but on the left side of diagrams.

Turning to functoriality axioms, we start with functoriality w.r.t. units. To this end, let us suppose an object c_0 of \mathbb{C} been given. Then

$$\begin{aligned} u(c_0) \bullet^0 [\alpha * \beta]_1^{c_0, c_0} &= u(c_0) \bullet^0 (\alpha^{c_0, c_0} * \beta_1^{Fc_0, Gc_0}) \\ \underline{(i)} & (u(c_0) \bullet^0 \alpha^{c_0, c_0}) * \beta_1^{Fc_0, Gc_0} \\ \underline{(ii)} & id_{[\alpha(c_0)]} * \beta_1^{Fc_0, Gc_0} \\ \underline{(iii)} & id_{[\alpha(c_0)]} \bullet^0 \beta_1^{Fc_0, Gc_0} \\ \underline{(iv)} & id_{[\beta_1^{Fc_0, Gc_0}](\alpha(c_0))} \\ &= id_{[\alpha * \beta]_0(c_0)} \end{aligned}$$

where first and last equation are definitions, (i) holds by $*$ -associativity, (ii) by units axioms for α , (iii) by $*$ -identity property, (iv) is simply the application of a 2-morphism to a constant 1-morphism.

To prove composition axiom, we start by fixing arbitrary objects c_0, c'_0, c''_0 of \mathbb{C} . Then it is easier to start from the result and back-track the chain of equalities as shown below. By $*$ -associativity one has

$$(- \circ -) \bullet^0 (\alpha_1^{c_0, c''_0} * \beta_1^{Fc_0, Gc''_0}) = ((- \circ -) \bullet^0 \alpha_1^{c_0, c''_0}) * \beta_1^{Fc_0, Gc''_0}$$

applying composition coherence of α

$$\left[\left(\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0} \right) \bullet^1 \left(F_1^{c_0, c'_0} \circ \alpha_1^{c'_0, c''_0} \right) \right] * \beta_1^{F c_0, G c''_0}$$

by *-functoriality

$$\begin{aligned} & \left[\left(\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0} \right) * \beta_1^{F c_0, G c''_0} \right] \bullet^1 \left[\left(F_1^{c_0, c'_0} \circ \alpha_1^{c'_0, c''_0} \right) \bullet^0 \left(H_1^{F c_0, G c''_0} \circ \beta_{G c''_0} \right) \right] \\ & \quad \bullet^2 \\ & \left[\left(\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0} \right) \bullet^0 \left(\beta_{F c_0} \circ K_1^{F c_0, G c''_0} \right) \right] \bullet^1 \left[\left(F_1^{c_0, c'_0} \circ \alpha_1^{c'_0, c''_0} \right) * \beta_1^{F c_0, G c''_0} \right] \end{aligned}$$

that is

$$\begin{aligned} & \left[\left(\left(\alpha_1^{c_0, c'_0} \times G_1^{c'_0, c''_0} \right) \bullet^0 (- \circ -) \right) * \beta_1^{F c_0, G c''_0} \right] \bullet^1 \left[\left(F_1^{c_0, c'_0} \circ \alpha_1^{c'_0, c''_0} \right) \bullet^0 \left(H_1^{F c_0, G c''_0} \circ \beta_{G c''_0} \right) \right] \\ & \quad \bullet^2 \\ & \left[\left(\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0} \right) \bullet^0 \left(\beta_{F c_0} \circ K_1^{F c_0, G c''_0} \right) \right] \bullet^1 \left[\left(\left(F_1^{c_0, c'_0} \times \alpha_1^{c'_0, c''_0} \right) \bullet^0 (- \circ -) \right) * \beta_1^{F c_0, G c''_0} \right] \end{aligned}$$

Then by *-associativity we obtain

$$\begin{aligned} & \left[\left(\alpha_1^{c_0, c'_0} \times G_1^{c'_0, c''_0} \right) * \left((- \circ -) \bullet^0 \beta_1^{F c_0, G c''_0} \right) \right] \bullet^1 \left[\left(F_1^{c_0, c'_0} \circ \alpha_1^{c'_0, c''_0} \right) \bullet^0 \left(H_1^{F c_0, G c''_0} \circ \beta_{G c''_0} \right) \right] \\ & \quad \bullet^2 \\ & \left[\left(\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0} \right) \bullet^0 \left(\beta_{F c_0} \circ K_1^{F c_0, G c''_0} \right) \right] \bullet^1 \left[\left(F_1^{c_0, c'_0} \times \alpha_1^{c'_0, c''_0} \right) * \left((- \circ -) \bullet^0 \beta_1^{F c_0, G c''_0} \right) \right] \end{aligned}$$

applying now composition axiom of β this turns in

$$\begin{aligned} & \left[\left(\alpha_1^{c_0, c'_0} \times G_1^{c'_0, c''_0} \right) * \left(\left(\beta_1^{F c_0, G c'_0} \circ K_1^{G c'_0, G c''_0} \right) \bullet^1 \left(H_1^{F c_0, G c'_0} \circ \beta_1^{G c'_0, G c''_0} \right) \right) \right] \bullet^1 \left[\left(F_1^{c_0, c'_0} \circ \alpha_1^{c'_0, c''_0} \right) \bullet^0 \left(H_1^{F c_0, G c''_0} \circ \beta_{G c''_0} \right) \right] \\ & \quad \bullet^2 \\ & \left[\left(\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0} \right) \bullet^0 \left(\beta_{F c_0} \circ K_1^{F c_0, G c''_0} \right) \right] \bullet^1 \left[\left(F_1^{c_0, c'_0} \times \alpha_1^{c'_0, c''_0} \right) * \left(\left(\beta_1^{F c_0, F c'_0} \circ K_1^{F c'_0, G c''_0} \right) \bullet^1 \left(H_1^{F c_0, F c'_0} \circ \beta_1^{F c'_0, G c''_0} \right) \right) \right] \end{aligned}$$

i.e.

$$\begin{aligned} & \left(\begin{array}{c} \left(\alpha_1^{c_0, c'_0} \times G_1^{c'_0, c''_0} \right) * \left(\left(\beta_1^{F c_0, G c'_0} \circ K_1^{G c'_0, G c''_0} \right) \bullet^1 \left(H_1^{F c_0, G c'_0} \circ \beta_1^{G c'_0, G c''_0} \right) \right) \\ \bullet^1 \\ \left(F_1^{c_0, c'_0} \circ \alpha_1^{c'_0, c''_0} \right) \bullet^0 \left(H_1^{F c_0, G c''_0} \circ \beta_{G c''_0} \right) \end{array} \right) \\ & \quad \bullet^2 \\ & \left(\begin{array}{c} \left(\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0} \right) \bullet^0 \left(\beta_{F c_0} \circ K_1^{F c_0, G c''_0} \right) \\ \bullet^1 \\ \left(F_1^{c_0, c'_0} \times \alpha_1^{c'_0, c''_0} \right) * \left(\left(\beta_1^{F c_0, F c'_0} \circ K_1^{F c'_0, G c''_0} \right) \bullet^1 \left(H_1^{F c_0, F c'_0} \circ \beta_1^{F c'_0, G c''_0} \right) \right) \end{array} \right) \end{aligned} \tag{5.6}$$

Let us focus our attention on the second composite (w.r.t. \bullet^2 -composition). In fact the calculations on the first component are precisely symmetrical.

Applying $*$ -functoriality to this we get

$$\begin{aligned} & \left(\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0} \right) \bullet^0 \left(\beta_{Fc_0} \circ K_1^{Fc_0, Gc''_0} \right) \\ & \bullet^1 \left(\left((F_1^{c_0, c'_0} \times \alpha_1^{c'_0, c''_0}) * (\beta_1^{Fc_0, Fc'_0} \circ K_1^{Fc'_0, Gc''_0}) \right) \bullet^1 \left((F_1^{c_0, c'_0} \times F_1^{c'_0, c''_0} \circ \alpha_{c'_0}) \bullet^0 (H_1^{Fc_0, Fc'_0} \circ \beta_1^{Fc'_0, Gc''_0}) \right) \right) \\ & \bullet^2 \left(\left((F_1^{c_0, c'_0} \times \alpha_{c'_0} \circ G_1^{c'_0, c''_0}) \bullet^0 (\beta_1^{Fc_0, Fc'_0} \circ K_1^{Fc'_0, Gc''_0}) \right) \bullet^1 \left((F_1^{c_0, c'_0} \times \alpha_1^{c'_0, c''_0}) * (H_1^{Fc_0, Fc'_0} \circ \beta_1^{Fc'_0, Gc''_0}) \right) \right) \end{aligned}$$

By composing on product components we notice that upper $*$ -composition gives indeed an identity

$$\begin{aligned} & \left(\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0} \right) \bullet^0 \left(\beta_{Fc_0} \circ K_1^{Fc_0, Gc''_0} \right) \\ & \bullet^1 \left(\left((F_1^{c_0, c'_0} \bullet^0 \beta_1^{Fc_0, Fc'_0}) \circ (\alpha_1^{c'_0, c''_0} \bullet^0 K_1^{Fc'_0, Gc''_0}) \right) \bullet^1 \left((F_1^{c_0, c'_0} \times F_1^{c'_0, c''_0} \circ \alpha_{c'_0}) \bullet^0 (H_1^{Fc_0, Fc'_0} \circ \beta_1^{Fc'_0, Gc''_0}) \right) \right) \\ & \bullet^2 \left(\left((F_1^{c_0, c'_0} \times \alpha_{c'_0} \circ G_1^{c'_0, c''_0}) \bullet^0 (\beta_1^{Fc_0, Fc'_0} \circ K_1^{Fc'_0, Gc''_0}) \right) \bullet^1 \left((F_1^{c_0, c'_0} \bullet^0 H_1^{Fc_0, Fc'_0}) \circ (\alpha_1^{c'_0, c''_0} * \beta_1^{Fc'_0, Gc''_0}) \right) \right) \end{aligned}$$

hence all the middle row is an identity 2-morphism, and the whole simplifies to the following

$$\begin{aligned} & \left(\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0} \right) \bullet^0 \left(\beta_{Fc_0} \circ K_1^{Fc_0, Gc''_0} \right) \\ & \bullet^1 \left((F_1^{c_0, c'_0} \times \alpha_{c'_0} \circ G_1^{c'_0, c''_0}) \bullet^0 (\beta_1^{Fc_0, Fc'_0} \circ K_1^{Fc'_0, Gc''_0}) \right) \bullet^1 \left((F_1^{c_0, c'_0} \bullet^0 H_1^{Fc_0, Fc'_0}) \circ (\alpha_1^{c'_0, c''_0} * \beta_1^{Fc'_0, Gc''_0}) \right) \end{aligned}$$

by \bullet^1 -whiskering associativity this can be rearranged

$$\begin{aligned} & \left((\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0}) \bullet^0 (\beta_{Fc_0} \circ K_1^{Fc_0, Gc''_0}) \right) \bullet^1 \left((F_1^{c_0, c'_0} \times \alpha_{c'_0} \circ G_1^{c'_0, c''_0}) \bullet^0 (\beta_1^{Fc_0, Fc'_0} \circ K_1^{Fc'_0, Gc''_0}) \right) \\ & \bullet^1 \left((F_1^{c_0, c'_0} \bullet^0 H_1^{Fc_0, Fc'_0}) \circ (\alpha_1^{c'_0, c''_0} * \beta_1^{Fc'_0, Gc''_0}) \right) \end{aligned}$$

By functoriality of K

$$\begin{aligned} & \left((\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0}) \bullet^0 (\beta_{F_{c_0}} \circ K_1^{F_{c_0}, G_{c'_0}}) \right) \bullet^1 \left((F_1^{c_0, c'_0} \times G_1^{c'_0, c''_0}) \bullet^0 (\beta_1^{F_{c_0}, F_{c'_0}} \circ K_1^{F_{c'_0}, G_{c'_0}} (\alpha_{c'_0}) \circ K_1^{G_{c'_0}, G_{c''_0}}) \right) \\ & \quad \bullet^1 \\ & (F_1^{c_0, c'_0} \bullet^0 H_1^{F_{c_0}, F_{c'_0}}) \circ (\alpha_1^{c'_0, c''_0} * \beta_1^{F_{c'_0}, G_{c''_0}}) \end{aligned}$$

by functoriality of $- \circ -$

$$\begin{aligned} & \left((\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0}) \bullet^0 (\beta_{F_{c_0}} \circ K_1^{F_{c_0}, G_{c'_0}}) \right) \bullet^1 \left((F_1^{c_0, c'_0} \bullet^0 \beta_1^{F_{c_0}, F_{c'_0}}) \circ K_1^{F_{c'_0}, G_{c'_0}} (\alpha_{c'_0}) \circ (G_1^{c'_0, c''_0} \bullet^0 K_1^{G_{c'_0}, G_{c''_0}}) \right) \\ & \quad \bullet^1 \\ & (F_1^{c_0, c'_0} \bullet^0 H_1^{F_{c_0}, F_{c'_0}}) \circ (\alpha_1^{c'_0, c''_0} * \beta_1^{F_{c'_0}, G_{c''_0}}) \end{aligned}$$

that is

$$\begin{aligned} & \left(\beta_{F_{c_0}} \circ ((\alpha_1^{c_0, c'_0} \circ G_1^{c'_0, c''_0}) \bullet^0 K_1^{F_{c_0}, G_{c'_0}}) \right) \bullet^1 \left((F_1^{c_0, c'_0} \bullet^0 \beta_1^{F_{c_0}, F_{c'_0}}) \circ K_1^{F_{c'_0}, G_{c'_0}} (\alpha_{c'_0}) \circ (G_1^{c'_0, c''_0} \bullet^0 K_1^{G_{c'_0}, G_{c''_0}}) \right) \\ & \quad \bullet^1 \\ & (F_1^{c_0, c'_0} \bullet^0 H_1^{F_{c_0}, F_{c'_0}}) \circ (\alpha_1^{c'_0, c''_0} * \beta_1^{F_{c'_0}, G_{c''_0}}) \end{aligned}$$

again by functoriality of K we can write the result as

$$\begin{aligned} & \beta_{F_{c_0}} \circ (\alpha_1^{c_0, c'_0} \bullet^0 K_1^{F_{c_0}, G_{c'_0}}) \circ (G_1^{c'_0, c''_0} \bullet^0 K_1^{G_{c'_0}, G_{c''_0}}) \\ & \quad \bullet^1 \\ & (F_1^{c_0, c'_0} \bullet^0 \beta_1^{F_{c_0}, F_{c'_0}}) \circ K_1^{F_{c'_0}, G_{c'_0}} (\alpha_{c'_0}) \circ (G_1^{c'_0, c''_0} \bullet^0 K_1^{G_{c'_0}, G_{c''_0}}) \\ & \quad \bullet^1 \\ & (F_1^{c_0, c'_0} \bullet^0 H_1^{F_{c_0}, F_{c'_0}}) \circ (\alpha_1^{c'_0, c''_0} * \beta_1^{F_{c'_0}, G_{c''_0}}) \end{aligned}$$

or more simply

$$\begin{aligned} & \left((\beta_{F_{c_0}} \circ (\alpha_1^{c_0, c'_0} \bullet^0 K_1^{F_{c_0}, G_{c'_0}})) \bullet^1 \left((F_1^{c_0, c'_0} \bullet^0 \beta_1^{F_{c_0}, F_{c'_0}}) \circ K_1^{F_{c'_0}, G_{c'_0}} (\alpha_{c'_0}) \right) \right) \circ (G_1^{c'_0, c''_0} \bullet^0 K_1^{G_{c'_0}, G_{c''_0}}) \\ & \quad \bullet^1 \\ & (F_1^{c_0, c'_0} \bullet^0 H_1^{F_{c_0}, F_{c'_0}}) \circ (\alpha_1^{c'_0, c''_0} * \beta_1^{F_{c'_0}, G_{c''_0}}) \end{aligned}$$

i.e.

$$\begin{aligned} & [(F \bullet^0 \beta) \bullet^1 (\alpha \bullet^0 K)]_1^{c_0, c'_0} \circ [G \bullet^0 K]_1^{c'_0, c''_0} \\ & \quad \bullet^1 \\ & [F \bullet^0 H]_1^{c_0, c'_0} \circ [\alpha * \beta]_1^{c'_0, c''_0} \end{aligned}$$

Carrying on the analogous calculations on the first component, (5.6) equals to

$$\begin{aligned} & \left([\alpha * \beta]_1^{c_0, c'_0} \circ [G \bullet^0 K]_1^{c'_0, c''_0} \right) \bullet^1 \left([F \bullet^0 H]_1^{c_0, c'_0} \circ [(F \bullet^0 \beta) \bullet^1 (\alpha \bullet^0 K)]_1^{c'_0, c''_0} \right) \\ & \quad \bullet^2 \\ & \left([(F \bullet^0 \beta) \bullet^1 (\alpha \bullet^0 K)]_1^{c_0, c'_0} \circ [G \bullet^0 K]_1^{c'_0, c''_0} \right) \bullet^1 \left([F \bullet^0 H]_1^{c_0, c'_0} \circ [\alpha * \beta]_1^{c'_0, c''_0} \right) \end{aligned}$$

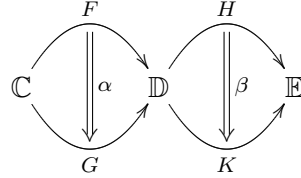
that is

$$\begin{aligned} & \left([\alpha * \beta]_1^{c_0, c'_0} \circ [G \bullet^0 K]_1^{c'_0, c''_0} \right) \bullet^1 \left([F \bullet^0 H]_1^{c_0, c'_0} \circ [\mathbf{dom}(\alpha * \beta)]_1^{c'_0, c''_0} \right) \\ & \quad \bullet^2 \\ & \left([\mathbf{cod}(\alpha * \beta)]_1^{c_0, c'_0} \circ [G \bullet^0 K]_1^{c'_0, c''_0} \right) \bullet^1 \left([F \bullet^0 H]_1^{c_0, c'_0} \circ [\alpha * \beta]_1^{c'_0, c''_0} \right) \end{aligned}$$

and this conclude the proof. \square

Remark 5.10. We have adopted the $*$ -symbol instead of the more obvious \bullet^0 in order to emphasize the dimension-raising property of this composition. Nevertheless $*$ -properties w.r.t. other \bullet^0 -compositions are somehow better understood thinking only in terms of \bullet^0 .

Lemma 5.11. *Given the case*



*If α is a lax natural n -transformation and β is a strict natural n -transformation, the composition $\alpha * \beta$ is an identity.*

In this case it is possible to deal with dimension preserving 0-composition of 2-morphisms, by letting

$$\alpha \tilde{*} \beta = \mathbf{dom}(\alpha * \beta) = \mathbf{cod}(\alpha * \beta)$$

Proof. Let us suppose α is a lax natural n -transformation and β is a strict natural n -transformation. Then for an object c_0 of \mathbb{C} , $\beta_1(\alpha_{c_0})$ is the commutative square $\beta_{Fc_0} \circ K(\alpha_{c_0}) = H(\alpha_{c_0}) \circ \beta_{Gc_0}$.

Moreover once objects c_0, c'_0 of \mathbb{C} are fixed, since β is strict

$$\beta_1^{Fc_0, Gc'_0} = id_{H_1^{Fc_0, Gc'_0} \circ \beta_{Gc'_0}} = id_{\alpha_{Fc_0} \circ K_1^{Fc_0, Gc'_0}}$$

then

$$\begin{aligned}
 [\alpha * \beta]_1^{c_0, c'_0} &= \alpha_1^{c_0, c'_0} * \beta_1^{F c_0, G c'_0} \\
 &= \alpha_1^{c_0, c'_0} * id_{H_1^{F c_0, G c'_0} \circ \beta_{G c'_0}} \\
 &= id_{\alpha_1^{c_0, c'_0} \bullet^0 (H_1^{F c_0, G c'_0} \circ \beta_{G c'_0})}
 \end{aligned}$$

Now the whiskering $\alpha_1^{c_0, c'_0} \bullet^0 (H_1^{F c_0, G c'_0} \circ \beta_{G c'_0})$ has domain

$$\begin{aligned}
 \mathbf{dom}(\alpha_1^{c_0, c'_0} \bullet^0 (H_1^{F c_0, G c'_0} \circ \beta_{G c'_0})) &= (F_1^{c_0, c'_0} \circ \alpha_{c'_0}) \bullet^0 (H_1^{F c_0, G c'_0} \circ \beta_{G c'_0}) \\
 &= (F_1^{c_0, c'_0} \bullet^0 H_1^{F c_0, G c'_0}) \circ H(\alpha_{c'_0}) \circ \beta_{G c'_0} \\
 &= [F \bullet^0 H]_1^{c_0, c'_0} \circ [\alpha \tilde{*} \beta]_{c'_0}
 \end{aligned}$$

Similarly the codomain is

$$\begin{aligned}
 \mathbf{cod}(\alpha_1^{c_0, c'_0} \bullet^0 (H_1^{F c_0, G c'_0} \circ \beta_{G c'_0})) &= (\alpha_{c_0} \circ G_1^{c_0, c'_0}) \bullet^0 (H_1^{F c_0, G c'_0} \circ \beta_{G c'_0}) \\
 &= (\alpha_{c_0} \circ G_1^{c_0, c'_0}) \bullet^0 (\beta_{F c_0} \circ K_1^{F c_0, G c'_0}) \\
 &= \beta_{F c_0} \circ K(\alpha_{c_0}) \circ (G_1^{c_0, c'_0} \bullet^0 K_1^{F c_0, G c'_0}) \\
 &= [\alpha \tilde{*} \beta]_{c_0} \circ [G \bullet^0 K]_1^{c_0, c'_0}
 \end{aligned}$$

hence the result is (an identity over) a 2-morphism. \square

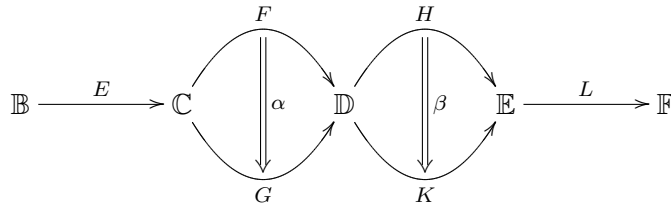
Importance of *Lemma* above is in that it allows to right-0-compose freely with constant transformations, such as $- \circ c_2$ or $c_2 \circ -$ for a 2-cell $c_2 : c_1 \Rightarrow c'_1 : c_0 \rightarrow c'_0$.

Notice that *Lemma* does not hold for α strict and β lax, since in this case the result is a strict 3-morphism.

5.7.1 Properties

The following propositions conclude the description of dimension-rising composition in the sesqui²category of strict n -categories.

Given the situation



one has the following

Proposition 5.12 (*-associativity 1).

$$(L * A) \quad (E \bullet_L^0 \alpha) * \beta = E \bullet_L^0 (\alpha * \beta) \quad (R * A) \quad \alpha * (\beta \bullet_R^0 L) = (\alpha * \beta) \bullet_R^0 L$$

Proof. We prove $(L * A)$. The proof of $(R * A)$ is similar hence it is omitted.

(on objects) Let an object b_0 of \mathbb{C} be given. Then

$$\begin{aligned} [(E \bullet^0 \alpha) * \beta]_{b_0} &= \beta([E \bullet^0 \alpha]_{b_0}) \\ &= \beta(\alpha_{Eb_0}) \\ &= [\alpha * \beta]_{Eb_0} \\ &= [E \bullet^0 (\alpha * \beta)]_{b_0} \end{aligned}$$

(on homs) Let objects b_0, b'_0 of \mathbb{B} be given. Then

$$\begin{aligned} [(E \bullet^0 \alpha) * \beta]_1^{b_0, b'_0} &= [E \bullet^0 \alpha]_1^{b_0, b'_0} * \beta_1^{F(Eb_0), G(Eb'_0)} \\ &= (E_1^{b_0, b'_0} \bullet^0 \alpha_1^{Eb_0, Eb'_0}) * \beta_1^{F(Eb_0), G(Eb'_0)} \\ &\stackrel{(\spadesuit)}{=} E_1^{b_0, b'_0} \bullet^0 (\alpha_1^{Eb_0, Eb'_0} * \beta_1^{F(Eb_0), G(Eb'_0)}) \\ &= E_1^{b_0, b'_0} \bullet^0 [\alpha_1 * \beta]^{Eb_0, Eb'_0} \\ &= [E \bullet^0 (\alpha * \beta)]_1^{b_0, b'_0} \end{aligned}$$

where (\spadesuit) holds by *-associativity in dimension $n - 1$. \square

Proposition 5.13 (*-identity).

$$(L) \quad id_E * \alpha = id_{E \bullet_L^0 \alpha} \quad (R) \quad \alpha * id_H = id_{\alpha \bullet_R^0 H}$$

Proof. We prove (L) . The proof of (R) is similar hence it is omitted.

(on objects) Let an object b_0 of \mathbb{B} be given. Then the following equalities are straightforward

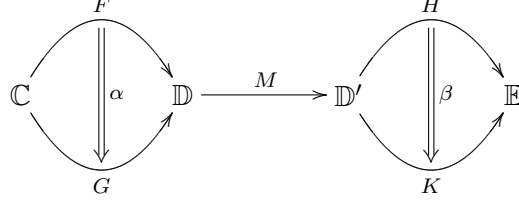
$$\begin{aligned} [id_E * \alpha]_{b_0} &= \alpha([id_E]_{b_0}) \\ &= \alpha(id_{Eb_0}) \\ &= id_{\alpha(Eb_0)} \\ &= [id_{E \bullet_L^0 \alpha}]_{b_0} \end{aligned}$$

(on homs) Let objects b_0, b'_0 of \mathbb{B} be given. Then

$$\begin{aligned} [id_E * \alpha]_1^{b_0, b'_0} &= [id_E]_1^{b_0, b'_0} * \alpha_1^{Eb_0, Eb'_0} \\ &= id_{E_1^{b_0, b'_0}} * \alpha_1^{Eb_0, Eb'_0} \\ &\stackrel{(\spadesuit)}{=} id_{E_1^{b_0, b'_0} \bullet^0 \alpha_1^{Eb_0, Eb'_0}} \\ &= [id_{E \bullet_L^0 \alpha}]_1^{b_0, b'_0} \end{aligned}$$

where (\spadesuit) is *-identity in dimension $n - 1$. \square

In the situation



one has the following

Proposition 5.14 (*-associativity 2).

$$\alpha * (M \bullet_L^0 \beta) = (\alpha \bullet_R^0 M) * \beta$$

Proof. (on objects) Let an object c_0 of \mathbb{C} be given. Then the following equalities are straightforward

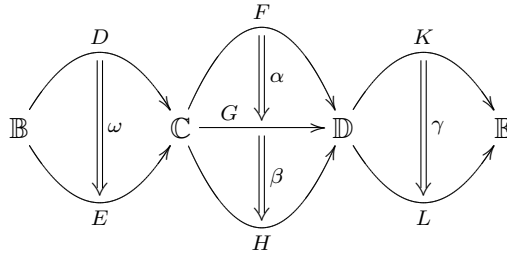
$$\begin{aligned} [\alpha * (M \bullet^0 \beta)]_{c_0} &= [M \bullet^0 \beta](\alpha_{c_0}) \\ &= \beta(M(\alpha_{c_0})) \\ &= \beta([\alpha \bullet^0 M]_{c_0}) \\ &= [(\alpha \bullet^0 M) * \beta]_{c_0} \end{aligned}$$

(on homs) Let objects c_0, c'_0 of \mathbb{C} be given. Then

$$\begin{aligned} [\alpha * (M \bullet^0 \beta)]_1^{c_0, c'_0} &= \alpha_1^{c_0, c'_0} * [M \bullet^0 \beta]_1^{Fc_0, Gc'_0} \\ &= \alpha_1^{c_0, c'_0} * \left(M_1^{Fc_0, Gc'_0} \bullet^0 \beta_1^{M(Fc_0), M(Gc'_0)} \right) \\ &\stackrel{(\spadesuit)}{=} \left(\alpha_1^{c_0, c'_0} \bullet^0 M_1^{Fc_0, Gc'_0} \right) * \beta_1^{M(Fc_0), M(Gc'_0)} \\ &= [\alpha \bullet^0 M]_1^{c_0, c'_0} * \beta_1^{M(Fc_0), M(Gc'_0)} \\ &= [(\alpha \bullet^0 M) * \beta]_1^{c_0, c'_0} \end{aligned}$$

where (\spadesuit) is *-associativity in dimension $n - 1$. □

In the situation below



one has the following

Proposition 5.15 (*-functoriality).

$$(a) \quad (\alpha \bullet^1 \beta) * \gamma = \left((\alpha * \gamma) \bullet^1 (\beta \bullet^0 L) \right) \bullet^2 \left((\alpha \bullet^0 K) \bullet^1 (\beta * \gamma) \right)$$

$$(b) \quad \omega * (\alpha \bullet^1 \beta) = \left((\omega * \alpha) \bullet^1 (E \bullet^0 \beta) \right) \bullet^2 \left((D \bullet^0 \alpha) \bullet^1 (\omega * \beta) \right)$$

Proof. We prove (a). The proof of (b) is similar, hence it is omitted.

(on objects) Let an object c_0 of \mathbb{C} be given. Then

$$\begin{aligned} \left[\begin{array}{c} (\alpha * \gamma) \bullet^1 (\beta \bullet^0 L) \\ \bullet^2 \\ (\alpha \bullet^0 K) \bullet^1 (\beta * \gamma) \end{array} \right]_{c_0} &= \begin{array}{c} [(\alpha * \gamma) \bullet^1 (\beta \bullet^0 L)]_{c_0} \\ \circ^1 \\ [(\alpha \bullet^0 K) \bullet^1 (\beta * \gamma)]_{c_0} \end{array} \\ &\stackrel{(\spadesuit)}{=} \begin{array}{c} \gamma(\alpha_{c_0}) \bullet^0 L(\beta_{c_0}) \\ \circ^1 \\ K(\alpha_{c_0}) \bullet^0 \gamma(\beta_{c_0}) \end{array} \\ &= \gamma(\alpha_{c_0} \circ \beta_{c_0}) \\ &= [(\alpha \bullet^1 \beta) * \gamma]_{c_0} \end{aligned}$$

where all the equalities are just definitions, but (\spadesuit) that is given by functoriality w.r.t. 0-composition of γ .

(on homs) Let objects c_0, c'_0 of \mathbb{C} be given. Applying the definition of 2-composition of 3-morphisms

$$\begin{aligned} &\left[\begin{array}{c} (\alpha * \gamma) \bullet^1 (\beta \bullet^0 L) \\ \bullet^2 \\ (\alpha \bullet^0 K) \bullet^1 (\beta * \gamma) \end{array} \right]_1^{c_0, c'_0} = \\ &= \left[(\alpha \bullet^0 K) \bullet^1 (\beta * \gamma) \right]_1^{c_0, c'_0} \bullet^2 \left([\alpha \bullet^0 K]_1^{c_0, c'_0} \circ \mathbf{cod}(\beta * \gamma) \right) \\ &= \left(\mathbf{dom}(\alpha * \gamma) \circ [\beta \bullet^0 L]_1^{c_0, c'_0} \right) \bullet^1 \left[(\alpha * \gamma) \bullet^1 (\beta \bullet^0 L) \right]_1^{c_0, c'_0} \end{aligned}$$

by definition of whiskering of 3-morphisms and 2-morphisms

$$\begin{aligned} &\left([\alpha \bullet^0 K]_{c_0} \circ [\beta * \gamma]_1^{c_0, c'_0} \right) \bullet^1 \left([\alpha \bullet^0 K]_1^{c_0, c'_0} \circ \mathbf{cod}(\beta * \gamma) \right) \\ &\bullet^2 \\ &\left(\mathbf{dom}(\alpha * \gamma) \circ [\beta \bullet^0 L]_1^{c_0, c'_0} \right) \bullet^1 \left([\alpha * \gamma]_1^{c_0, c'_0} \circ [\beta \bullet^0 L]_{c'_0} \right) \end{aligned}$$

that is

$$\begin{aligned} &\left([\alpha \bullet^0 K]_{c_0} \circ [\beta * \gamma]_1^{c_0, c'_0} \right) \bullet^1 \left(\left([\alpha \bullet^0 K]_1^{c_0, c'_0} \right) \circ K(\beta_{c'_0}) \circ \gamma_{Hc'_0} \right) \\ &\bullet^2 \\ &\left(\gamma_{Fc_0} \circ L(\alpha_{c_0}) \circ [\beta \bullet^0 L]_1^{c_0, c'_0} \right) \bullet^1 \left([\alpha * \gamma]_1^{c_0, c'_0} \circ [\beta \bullet^0 L]_{c'_0} \right) \end{aligned}$$

by definition of $*$ -composition on homs (and of 0-whiskering for 2-morphisms)

$$\begin{aligned} & \left(K(\alpha_{c_0}) \circ \left(\beta_1^{c_0, c'_0} * \gamma_1^{Gc_0, Hc'_0} \right) \right) \bullet^1 \left(\left(\alpha_1^{c_0, c'_0} \bullet^0 K_1^{Fc_0, Gc'_0} \right) \circ K(\beta_{c'_0}) \circ \gamma_{Hc'_0} \right) \\ & \quad \bullet^2 \\ & \left(\gamma_{Fc_0} \circ L(\alpha_{c_0}) \circ \left(\beta_1^{c_0, c'_0} \bullet^0 L_1^{Gc_0, Hc'_0} \right) \right) \bullet^1 \left(\left(\alpha_1^{c_0, c'_0} * \gamma_1^{Fc_0, Gc'_0} \right) \circ L(\beta_{c'_0}) \right) \end{aligned}$$

this can be rearranged

$$\begin{aligned} & \left(K(\alpha_{c_0}) \circ \left(\beta_1^{c_0, c'_0} * \gamma_1^{Gc_0, Hc'_0} \right) \right) \bullet^1 \left(\alpha_1^{c_0, c'_0} \bullet^0 \left(K_1^{Fc_0, Gc'_0} \circ K(\beta_{c'_0}) \circ \gamma_{Hc'_0} \right) \right) \\ & \quad \bullet^2 \\ & \left(\beta_1^{c_0, c'_0} \bullet^0 \left(\gamma_{Fc_0} \circ L(\alpha_{c_0}) \circ L_1^{Gc_0, Hc'_0} \right) \right) \bullet^1 \left(\left(\alpha_1^{c_0, c'_0} * \gamma_1^{Fc_0, Gc'_0} \right) \circ L(\beta_{c'_0}) \right) \end{aligned}$$

by functoriality of L and K

$$\begin{aligned} & \left(K(\alpha_{c_0}) \circ \left(\beta_1^{c_0, c'_0} * \gamma_1^{Gc_0, Hc'_0} \right) \right) \bullet^1 \left(\left(\alpha_1^{c_0, c'_0} \circ \beta_{c'_0} \right) \bullet^0 \left(K_1^{Fc_0, Hc'_0} \circ \gamma_{Hc'_0} \right) \right) \\ & \quad \bullet^2 \\ & \left(\left(\alpha_{c_0} \circ \beta_1^{c_0, c'_0} \right) \bullet^0 \left(\gamma_{Fc_0} \circ L_1^{Fc_0, Hc'_0} \right) \right) \bullet^1 \left(\left(\alpha_1^{c_0, c'_0} * \gamma_1^{Fc_0, Gc'_0} \right) \circ L(\beta_{c'_0}) \right) \end{aligned}$$

by functoriality (and $*$ -associativity)

$$\begin{aligned} & \left(\left(\alpha_{c_0} \circ \beta_1^{c_0, c'_0} \right) * \gamma_1^{Fc_0, Hc'_0} \right) \bullet^1 \left(\left(\alpha_1^{c_0, c'_0} \circ \beta_{c'_0} \right) \bullet^0 \left(K_1^{Fc_0, Hc'_0} \circ \gamma_{Hc'_0} \right) \right) \\ & \quad \bullet^2 \\ & \left(\left(\alpha_{c_0} \circ \beta_1^{c_0, c'_0} \right) \bullet^0 \left(\gamma_{Fc_0} \circ L_1^{Fc_0, Hc'_0} \right) \right) \bullet^1 \left(\left(\alpha_1^{c_0, c'_0} \circ \beta_{c'_0} \right) * \gamma_1^{Fc_0, Hc'_0} \right) \end{aligned}$$

and finally by $*$ -functoriality

$$\left(\left(\alpha_{c_0} \circ \beta_1^{c_0, c'_0} \right) \bullet^1 \left(\alpha_1^{c_0, c'_0} \circ \beta_{c'_0} \right) \right) * \gamma_1^{Fc_0, Hc'_0}$$

that is the result:

$$\left[(\alpha \bullet^1 \beta) * \gamma \right]_1^{c_0, c'_0}.$$

□

Chapter 6

h-Pullbacks revisited and the long exact sequence

6.1 2-dimensional *h*-pullbacks in $n\mathbf{Cat}$

We introduce here a notion of 2-dimensional *h*-pullback in the sesqui²-category $n\mathbf{Cat}$. It will be shown that our construction of the *standard h*-pullback of n -categories is an instance of such a 2-dimensional one.

In order to fix notation, let us consider the following diagram in $n\mathbf{Cat}$

$$\begin{array}{ccc} & \mathbb{C} & \\ & \downarrow G & \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array}$$

A *h*-2pullback of F and G is a four-tuple $(\mathbb{P}, P, Q, \varepsilon)$

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{Q} & \mathbb{C} \\ P \downarrow & \nearrow \varepsilon & \downarrow G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array}$$

that satisfies the following 2-dimensional universal property:

Universal Property 6.1 (*h*-2pullbacks). *For any other two four-tuple*

$$\begin{array}{ccc}
(\mathbb{X}, M, N, \omega) & & (\mathbb{X}, \hat{M}, \hat{N}, \hat{\omega}) \\
\begin{array}{ccc} \mathbb{X} & \xrightarrow{N} & \mathbb{C} \\ M \downarrow & \nearrow \omega & \downarrow G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array} & \text{and} & \begin{array}{ccc} \mathbb{X} & \xrightarrow{\hat{N}} & \mathbb{C} \\ \hat{M} \downarrow & \nearrow \hat{\omega} & \downarrow G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array} \\
2\text{-morphism } \alpha, \beta & & 3\text{-morphism } \Sigma \\
\begin{array}{ccccc} & & X & & \\ M \swarrow & & & \searrow N & \\ \mathbb{A} & \xleftarrow{\alpha} & \hat{M} & \xrightarrow{\beta} & \hat{N} & \searrow & \mathbb{C} \\ & & & & & & \end{array} & \text{and} & \begin{array}{ccc} M \bullet^0 F & \xrightarrow{\alpha \bullet^0 F} & \hat{M} \bullet^0 F \\ \omega \downarrow & \nearrow \Sigma & \downarrow \hat{\omega} \\ N \bullet^0 G & \xrightarrow{\beta \bullet^0 G} & \hat{N} \bullet^0 G \end{array}
\end{array}$$

there exists a unique $\lambda : L \Rightarrow \hat{L} : \mathbb{X} \rightarrow \mathbb{P}$ such that (UP)

1. $\lambda \bullet^0 P = \alpha$
2. $\lambda \bullet^0 Q = \beta$
3. $\lambda * \varepsilon = \Sigma$

As an immediate consequence of the definition, we state the following

Proposition 6.2. 2-Universal Property of h-2pullbacks implies 1-dimensional one. Hence h-2pullbacks are defined up to isomorphism.

Proof. Just put α, β and Σ identities. □

Let us notice that *Proposition 6.2* holds in every sesqui²-category. More interestingly in $n\mathbf{Cat}$ a kind of converse to this proposition also holds.

Proposition 6.3. Given the diagram

$$\begin{array}{ccc} & & \mathbb{C} \\ & & \downarrow G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array}$$

in $n\mathbf{Cat}$, the standard h-pullback $\langle \mathbb{P}, P, Q, \varepsilon \rangle$ satisfies also Universal property 6.1.

Proof. Firstly we remark that 1-dimensional *Universal Property 2.12* of *h*-pullbacks applied to the four-tuple $(\mathbb{X}, M, N, \omega)$ yields an $L : \mathbb{X} \rightarrow \mathbb{P}$, while applied to $(\mathbb{X}, \hat{M}, \hat{N}, \hat{\omega})$, a $\hat{L} : \mathbb{X} \rightarrow \mathbb{P}$. Those have to be *domain* and *co-domain* of the 2-cell provided by the universal property, namely $\lambda : L \Rightarrow \hat{L}$.

We recall the constructions in order to fix notation.

- $L_0 : \mathbb{X}_0 \rightarrow \mathbb{P}_0$ is the map

$$x_0 \mapsto (Mx_0, F(Mx_0) \xrightarrow{\omega(x_0)} G(Nx_0), Nx_0) =: p_0$$

- for every pair of objects x_0, x'_0 of \mathbb{X} ,

$$L_1^{x_0, x'_0} : \mathbb{X}_1(x_0, x'_0) \rightarrow \mathbb{P}_1(L(x_0), L(x'_0))$$

is given by the universal property in dimension $n - 1$, and is such that

$$L_1^{x_0, x'_0} \bullet^0 P_1^{Lx_0, Lx'_0} = M_1^{x_0, x'_0}$$

$$L_1^{x_0, x'_0} \bullet^0 Q_1^{Lx_0, Lx'_0} = N_1^{x_0, x'_0}$$

$$L_1^{x_0, x'_0} \bullet^0 \varepsilon_1^{Lx_0, Lx'_0} = \omega_1^{x_0, x'_0}$$

The pair $L = \langle L_0, L_1^{-, -} \rangle$ is a 1-morphism.

Similarly one determines $\hat{L} = \langle \hat{L}_0, \hat{L}_1^{-, -} \rangle$.

Now we show that remaining data (namely, α, β and Σ) of the hypothesis provide a 2-morphism $\lambda : L \Rightarrow \hat{L}$ that satisfies required property. To this end, let us consider the following assignments:

- For every object x_0 of \mathbb{X} ,

$$\lambda_{x_0} = (\alpha_{x_0}, \Sigma_{x_0}, \beta_{x_0}) : Lx_0 \rightarrow \hat{L}x_0$$

i.e.

$$\begin{array}{ccc} Mx_0 & \xrightarrow{\alpha_{x_0}} & \hat{M}x_0 \\ F(Mx_0) & \xrightarrow{F(\alpha_{x_0})} & F(\hat{M}x_0) \\ \omega_{x_0} \downarrow & \nearrow \Sigma_{x_0} & \downarrow \hat{\omega}_{x_0} \\ G(Nx_0) & \xrightarrow{G(\beta_{x_0})} & G(\hat{N})x_0 \\ Nx_0 & \xrightarrow{\beta_{x_0}} & \hat{N}x_0 \end{array}$$

- For every pair of objects x_0, x'_0 of \mathbb{X} ,

$$\begin{array}{ccc}
 & \mathbb{X}_1(x_0, x'_0) & \\
 L_1^{x_0, x'_0} \swarrow & & \searrow \hat{L}_1^{x_0, x'_0} \\
 \mathbb{P}_1(Lx_0, Lx'_0) & \xleftarrow{\lambda_1^{x_0, x'_0}} & \mathbb{P}_1(\hat{L}x_0, \hat{L}x'_0) \\
 \downarrow -\circ \lambda x'_0 & & \downarrow \lambda x_0 \circ - \\
 & \mathbb{P}_1(Lx_0, \hat{L}x'_0) &
 \end{array}$$

is given by the universal property for $(n-1)$ categories.

In fact the 0-codomain of $\lambda_1^{x_0, x'_0}$, namely $\mathbb{P}_1(Lx_0, \hat{L}x'_0)$ is defined inductively as a h -2pullback in $(n-1)\mathbf{Cat}$:

$$\begin{array}{ccccc}
 \mathbb{P}_1(Lx_0, \hat{L}x'_0) & \xrightarrow{Q_1^{Lx_0, \hat{L}x'_0}} & \mathbb{C}_1(Nx_0, \hat{N}x'_0) & & \\
 \downarrow P_1^{Lx_0, \hat{L}x'_0} & & \downarrow G_1^{Nx_0, \hat{N}x'_0} & & \\
 & \swarrow \varepsilon_1^{Lx_0, \hat{L}x'_0} & \mathbb{B}_1(G(Nx_0), G(\hat{N}x'_0)) & & \\
 \mathbb{A}_1(Mx_0, \hat{M}x'_0) & \xrightarrow{F_1^{Mx_0, \hat{M}x'_0}} & \mathbb{B}_1(F(Mx_0), F(\hat{M}x'_0)) & \xrightarrow{-\circ \hat{\omega} x'_0} & \mathbb{B}_1(F(Mx_0), G(\hat{N}x'_0)) \\
 & & & & \downarrow \omega x_0 \circ -
 \end{array}$$

Over the same base are also defined

$$\begin{array}{ccccc}
 \mathbb{X}_1(x_0, x'_0) & \xrightarrow{N_1^{x_0, x'_0} \circ \beta x'_0} & \mathbb{C}_1(Nx_0, \hat{N}x'_0) & & \\
 \downarrow M_1^{x_0, x'_0} \circ \alpha x'_0 & & \downarrow G_1^{Nx_0, \hat{N}x'_0} & & \\
 & \swarrow \theta = (\omega_1^{x_0, x'_0} \circ G(\beta x'_0)) \bullet^1 ([MF]_1^{x_0, x'_0} \circ \Sigma x'_0) & \mathbb{B}_1(G(Nx_0), G(\hat{N}x'_0)) & & \\
 \mathbb{A}_1(Mx_0, \hat{M}x'_0) & \xrightarrow{F_1^{Mx_0, \hat{M}x'_0}} & \mathbb{B}_1(F(Mx_0), F(\hat{M}x'_0)) & \xrightarrow{-\circ \hat{\omega} x'_0} & \mathbb{B}_1(F(Mx_0), G(\hat{N}x'_0)) \\
 & & & & \downarrow \omega x_0 \circ -
 \end{array}$$

and

$$\begin{array}{ccc}
 \mathbb{X}_1(x_0, x'_0) & \xrightarrow{\beta x_0 \circ \hat{N}_1^{x_0, x'_0}} & \mathbb{C}_1(Nx_0, \hat{N}x'_0) \\
 \downarrow \alpha x_0 \circ \hat{M}_1^{x_0, x'_0} & \nearrow \hat{\theta} = (\Sigma x_0 \circ [\hat{N}G]_1^{x_0, x'_0}) \bullet^1 (F(\alpha x_0) \circ \hat{\omega}_1^{x_0, x'_0}) & \downarrow G_1^{Nx_0, \hat{N}x'_0} \\
 & & \mathbb{B}_1(G(Nx_0), G(\hat{N}x'_0)) \\
 & & \downarrow \omega x_0 \circ - \\
 \mathbb{A}_1(Mx_0, \hat{M}x'_0) & \xrightarrow{F_1^{Mx_0, \hat{M}x'_0}} \mathbb{B}_1(F(Mx_0), F(\hat{M}x'_0)) \xrightarrow{- \circ \hat{\omega} x'_0} \mathbb{B}_1(F(Mx_0), G(\hat{N}x'_0)) &
 \end{array}$$

Moreover we can consider 2-morphisms:

$$\alpha_1^{x_0, x'_0} : \alpha x_0 \circ \hat{M}_1^{x_0, x'_0} \Rightarrow M_1^{x_0, x'_0} \circ \alpha x'_0 : \mathbb{X}_1(x_0, x'_0) \rightarrow \mathbb{A}_1(Mx_0, \hat{M}x'_0)$$

$$\beta_1^{x_0, x'_0} : \beta x_0 \circ \hat{N}_1^{x_0, x'_0} \Rightarrow N_1^{x_0, x'_0} \circ \beta x'_0 : \mathbb{X}_1(x_0, x'_0) \rightarrow \mathbb{A}_1(Mx_0, \hat{M}x'_0)$$

and the 3-morphism

$$\begin{array}{ccc}
 (\beta x_0 \circ \hat{N}_1^{x_0, x'_0}) \bullet^0 (\omega x_0 \circ G_1^{Nx_0, \hat{N}x'_0}) & \xrightarrow{\beta_1^{x_0, x'_0} \bullet^0 id} & (N_1^{x_0, x'_0} \circ \beta x'_0) \bullet^0 (\omega x_0 \circ G_1^{Nx_0, \hat{N}x'_0}) \\
 \Downarrow \hat{\theta} & \nearrow \Sigma_1^{x_0, x'_0} & \Downarrow \theta \\
 (\alpha x_0 \circ \hat{M}_1^{x_0, x'_0}) \bullet^0 (F_1^{Mx_0, \hat{M}x'_0} \circ \hat{\omega} x'_0) & \xrightarrow{\beta_1^{x_0, x'_0} \bullet^0 id} & (M_1^{x_0, x'_0} \circ \alpha x'_0) \bullet^0 (F_1^{Mx_0, \hat{M}x'_0} \circ \hat{\omega} x'_0)
 \end{array}$$

Finally we can apply the universal property, in order to get a *unique* 2-morphism

$$\lambda_1^{x_0, x'_0} : L_1^{x_0, x'_0} \circ \lambda x'_0 \Rightarrow \lambda x_0 \circ \hat{L}_1^{x_0, x'_0}$$

such that

$$\lambda_1^{x_0, x'_0} \bullet^0 Q_1^{Lx_0, \hat{L}x'_0} = \beta_1^{x_0, x'_0} \quad (6.1)$$

$$\lambda_1^{x_0, x'_0} \bullet^0 P_1^{Lx_0, \hat{L}x'_0} = \alpha_1^{x_0, x'_0} \quad (6.2)$$

$$\lambda_1^{x_0, x'_0} * \varepsilon_1^{Lx_0, \hat{L}x'_0} = \Sigma_1^{x_0, x'_0} \quad (6.3)$$

That the pair $\lambda = \langle \lambda_0, \lambda_1^{-, -} \rangle$ is a 2-morphism of n -categories is proved in the following (quite technical) *Lemma 6.4*.

Moreover it satisfies by construction *Universal Property 6.1*. In fact for any object x_0 of \mathbb{X}

$$[\lambda \bullet^0 P]_{x_0} = P(\lambda_{x_0}) = P((\alpha_{x_0}, \Sigma_{x_0}, \beta_{x_0})) = \alpha_{x_0}$$

and for any pair of x_0, x'_0

$$[\lambda \bullet^0 P]_1^{x_0, x'_0} = \lambda_1^{x_0, x'_0} \bullet^0 P_1^{Lx_0, \hat{L}x'_0} = \alpha_1^{x_0, x'_0}$$

thus $\lambda \bullet^0 P = \alpha$.

Similarly

$$[\lambda \bullet^0 Q]_{x_0} = Q(\lambda_{x_0}) = Q((\alpha_{x_0}, \Sigma_{x_0}, \beta_{x_0})) = \beta_{x_0}$$

and

$$[\lambda \bullet^0 Q]_1^{x_0, x'_0} = \lambda_1^{x_0, x'_0} \bullet^0 Q_1^{Lx_0, \hat{L}x'_0} = \beta_1^{x_0, x'_0}$$

thus $\lambda \bullet^0 Q = \beta$.

Finally

$$[\lambda * \varepsilon]_{x_0} = \varepsilon(\lambda_{x_0}) = \varepsilon((\alpha_{x_0}, \Sigma_{x_0}, \beta_{x_0})) = \Sigma_{x_0}$$

and

$$[\lambda * \varepsilon]_1^{x_0, x'_0} = \lambda_1^{x_0, x'_0} * \varepsilon_1^{Lx_0, \hat{L}x'_0} = \Sigma_1^{x_0, x'_0}$$

To conclude the proof we still need to prove uniqueness. But this will easily be achieved. Indeed the object part of 2-morphism λ satisfying the universal property is univocally determined by the fact that P_0, Q_0 and ε_0 are projection, and once that is determined, uniqueness in dimension $n - 1$ guaranties the homs part. \square

Lemma 6.4. *The pair $\lambda = \langle \lambda_0, \lambda_1^{-,-} \rangle$ is indeed a 2-morphism.*

Proof. We have to show that functoriality axioms for 2-morphisms are satisfied. Let us start with units axiom.

To this end let us fix an arbitrary object x_0 of \mathbb{X} . If we denote by $u(x_0)$ the identity $\mathbb{I}_{(n-1)} \rightarrow \mathbb{X}_1(x_0, x_0)$ then

$$\begin{aligned} u(x_0) \bullet^0 \lambda_1^{x_0, x_0} \bullet^0 Q_1^{Lx_0, \hat{L}x_0} & \stackrel{(i)}{=} u(x_0) \bullet^0 \beta_1^{x_0, x_0} \\ & \stackrel{(ii)}{=} id_{[\beta_{x_0}]} \\ & \stackrel{(iii)}{=} id_{[Q(\lambda_{x_0})]} \\ & \stackrel{(iv)}{=} id_{[\lambda_{x_0}]} \bullet^0 Q_1^{Lx_0, \hat{L}x_0} \end{aligned}$$

where (i) holds by (6.1) above, (ii) by unit functoriality of β , (iii) by definition of λ_0 , (iv) is a whiskering identity axiom. Similarly one can prove

$$u(x_0) \bullet^0 \lambda_1^{x_0, x_0} \bullet^0 P_1^{Lx_0, \hat{L}x_0} = id_{[\lambda_{x_0}]} \bullet^0 P_1^{Lx_0, \hat{L}x_0}$$

Moreover, for the formally analogous property w.r.t $*$ -composition

$$\begin{aligned}
 u(x_0) \bullet^0 \lambda_1^{x_0, x_0} * \varepsilon_1^{Lx_0, \hat{L}x_0} &= u(x_0) \bullet^0 \Sigma_1^{x_0, x_0} \\
 &= id_{[\Sigma x_0]} \\
 &= id_{[\varepsilon(\lambda x_0)]} \\
 &= id_{[\lambda x_0]} * \varepsilon_1^{Lx_0, \hat{L}x_0}
 \end{aligned}$$

Notice that first composites are unambiguous by associativity axioms. Calculations show that both $id_{[\lambda x_0]}$ and $u(x_0) \bullet^0 \lambda_1^{x_0, x_0}$ satisfy equations prescribed by the universal property, hence by uniqueness they must be equal, and unit axiom is proved.

Turning to composition coherence, let objects x_0, x'_0, x''_0 be given. Then

$$\begin{aligned}
 & \left((\lambda_1^{x_0, x'_0} \circ \hat{L}_1^{x'_0, x''_0}) \bullet^1 (L_1^{x_0, x'_0} \circ \lambda_1^{x'_0, x''_0}) \right) \bullet^0 Q_1^{Lx_0, \hat{L}x''_0} = \\
 \underline{(i)} \quad & \left((\lambda_1^{x_0, x'_0} \circ \hat{L}_1^{x'_0, x''_0}) \bullet^0 Q_1^{Lx_0, \hat{L}x''_0} \right) \bullet^1 \left((L_1^{x_0, x'_0} \circ \lambda_1^{x'_0, x''_0}) \bullet^0 Q_1^{Lx_0, \hat{L}x''_0} \right) \\
 \underline{(ii)} \quad & \left((\lambda_1^{x_0, x'_0} \bullet^0 Q_1^{Lx_0, \hat{L}x'_0}) \circ (\hat{L}_1^{x'_0, x''_0} \bullet^0 Q_1^{\hat{L}x'_0, \hat{L}x''_0}) \right) \bullet^1 \\
 & \bullet^1 \left((L_1^{x_0, x'_0} \bullet^0 Q_1^{Lx_0, Lx'_0}) \circ (\lambda_1^{x'_0, x''_0} \bullet^0 Q_1^{Lx'_0, \hat{L}x''_0}) \right) \\
 \underline{(iii)} \quad & (\beta_1^{x_0, x'_0} \circ \hat{N}_1^{x'_0, x''_0}) \bullet^1 (N_1^{x_0, x'_0} \circ \beta_1^{x'_0, x''_0}) \\
 \underline{(iv)} \quad & (- \circ -) \bullet^0 \beta_1^{x_0, x''_0} \\
 \underline{(v)} \quad & (- \circ -) \bullet^0 \lambda_1^{x_0, x''_0} \bullet^0 Q_1^{x_0, x''_0}
 \end{aligned}$$

where (i) holds by (sesqui)functoriality of $- \bullet^0 Q_1^{Lx_0, \hat{L}x''_0}$, (ii) by functoriality w.r.t. \circ -composition of Q , (iii) and (v) by (6.1), (iv) by functoriality w.r.t. composition of β .

Similarly one can prove

$$\left((\lambda_1^{x_0, x'_0} \circ \hat{L}_1^{x'_0, x''_0}) \bullet^1 (L_1^{x_0, x'_0} \circ \lambda_1^{x'_0, x''_0}) \right) \bullet^0 P_1^{Lx_0, \hat{L}x''_0} = (- \circ -) \bullet^0 \lambda_1^{x_0, x''_0} \bullet^0 P_1^{x_0, x''_0}$$

Now let us compute

$$\left((\lambda_1^{x_0, x'_0} \circ \hat{L}_1^{x'_0, x''_0}) \bullet^1 (L_1^{x_0, x'_0} \circ \lambda_1^{x'_0, x''_0}) \right) * \varepsilon_1^{Lx_0, \hat{L}x''_0}$$

by $*$ -functoriality this equals to

$$\begin{aligned}
& \underbrace{\left((\lambda_1^{x_0, x'_0} \circ \hat{L}_1^{x'_0, x''_0}) * \varepsilon^{Lx_0, \hat{L}x''_0} \right) \bullet^1 \left((L_1^{x_0, x'_0} \circ \lambda_1^{x'_0, x''_0}) \bullet^0 ([PF]_1^{Lx_0, \hat{L}x''_0} \circ \hat{\omega}x''_0) \right)}_{\bullet^2} \\
& \left((\lambda_1^{x_0, x'_0} \circ \hat{L}_1^{x'_0, x''_0}) \bullet^0 (\omega_{x_0} \circ [QG]_1^{Lx_0, \hat{L}x''_0}) \right) \bullet^1 \left((L_1^{x_0, x'_0} \circ \lambda_1^{x'_0, x''_0}) * \varepsilon_1^{Lx_0, \hat{L}x''_0} \right)
\end{aligned} \tag{6.4}$$

In order to simplify the expression, let us analyze first under-braced one. This can be re-written explicitly and processed by $*$ -associativity

$$\begin{aligned}
& (\lambda_1^{x_0, x'_0} \circ \hat{L}_1^{x'_0, x''_0}) * \varepsilon^{Lx_0, \hat{L}x''_0} = \\
& = \left((\lambda_1^{x_0, x'_0} \times \hat{L}_1^{x'_0, x''_0}) \bullet^0 (- \circ -) \right) * \varepsilon^{Lx_0, \hat{L}x''_0} \\
& = (\lambda_1^{x_0, x'_0} \times \hat{L}_1^{x'_0, x''_0}) * \left((- \circ -) \bullet^0 \varepsilon^{Lx_0, \hat{L}x''_0} \right) \\
& = (\lambda_1^{x_0, x'_0} \times \hat{L}_1^{x'_0, x''_0}) * \left((\varepsilon_1^{Lx_0, \hat{L}x'_0} \circ [QG]_1^{\hat{L}x'_0, \hat{L}x''_0}) \bullet^1 ([PF]_1^{Lx_0, \hat{L}x'_0} \circ \varepsilon_1^{\hat{L}x'_0, \hat{L}x''_0}) \right)
\end{aligned}$$

where the last is given by composition coherence of ε .

Applying again $*$ -functoriality, this turns to be

$$\begin{aligned}
& \left((\lambda_1^{x_0, x'_0} \times \hat{L}_1^{x'_0, x''_0}) * (\varepsilon_1^{Lx_0, \hat{L}x'_0} \circ [QG]_1^{\hat{L}x'_0, \hat{L}x''_0}) \right) \bullet^1 \left(((L_1^{x_0, x'_0} \circ \lambda_{x'_0}) \times \hat{L}_1^{x'_0, x''_0}) \bullet^0 ([PF]_1^{Lx_0, \hat{L}x'_0} \circ \varepsilon_1^{\hat{L}x'_0, \hat{L}x''_0}) \right) \\
& \bullet^2 \\
& \left(((\lambda_{x_0} \circ \hat{L}_1^{x_0, x'_0}) \times \hat{L}_1^{x'_0, x''_0}) \bullet^0 (\varepsilon_1^{Lx_0, \hat{L}x'_0} \circ [QG]_1^{\hat{L}x'_0, \hat{L}x''_0}) \right) \bullet^1 \left((\lambda_1^{x_0, x'_0} \times \hat{L}_1^{x'_0, x''_0}) * ([PF]_1^{Lx_0, \hat{L}x'_0} \circ \varepsilon_1^{\hat{L}x'_0, \hat{L}x''_0}) \right)
\end{aligned}$$

Now, all the second row is clearly an identity 3-morphism, being the $*$ -composition on separate components of a product, hence it can be canceled. What remains can be re-written as

$$\left((\lambda_1^{x_0, x'_0} * \varepsilon_1^{Lx_0, \hat{L}x'_0}) \circ [\hat{L}QG]_1^{x'_0, x''_0} \right) \bullet^1 \left([LPF]_1^{x_0, x'_0} \circ F(P(\lambda_{x'_0})) \circ (\hat{L}_1^{x'_0, x''_0} \bullet^0 \varepsilon_1^{\hat{L}x'_0, \hat{L}x''_0}) \right)$$

and with the help of (6.3) this is simply

$$(\Sigma_1^{x_0, x'_0} \circ [\hat{N}G]_1^{x'_0, x''_0}) \bullet^1 ([MF]_1^{x_0, x'_0} \circ F(P(\lambda_{x'_0})) \circ \hat{\omega}_1^{x'_0, x''_0})$$

Substituting the first line of (6.4) becomes a triple \bullet^1 -composition

$$\left(\Sigma_1^{x_0, x'_0} \circ [\hat{N}G]_1^{x'_0, x''_0} \right) \bullet^1 \left([MF]_1^{x_0, x'_0} \circ F(P(\lambda_{x'_0})) \circ \hat{\omega}_1^{x'_0, x''_0} \right) \bullet^1 \left((L_1^{x_0, x'_0} \circ \lambda_1^{x'_0, x''_0}) \bullet^0 ([PF]_1^{Lx_0, \hat{L}x''_0} \circ \hat{\omega}x''_0) \right) \tag{6.5}$$

Now $F(P(\lambda_{x'_0})) = F(\alpha_{x'_0}) = [\alpha \bullet^0 F]_{x'_0}$, furthermore $PF(L_1^{x_0, x'_0} \circ \lambda_1^{x'_0, x''_0}) = PF(L_1^{x_0, x'_0}) \circ PF(\lambda_1^{x'_0, x''_0}) = [MF]_1^{x_0, x'_0} \circ [\alpha \bullet^0 F]_1^{x'_0, x''_0}$. Hence (6.5) is equal to

$$\left(\Sigma_1^{x_0, x'_0} \circ [\hat{N}G]_1^{x'_0, x''_0} \right) \bullet^1 \left([MF]_1^{x_0, x'_0} \circ [\alpha \bullet^0 F]_{x'_0} \circ \hat{\omega}_1^{x'_0, x''_0} \right) \bullet^1 \left([MF]_1^{x_0, x'_0} \circ [\alpha \bullet^0 F]_1^{x'_0, x''_0} \circ \hat{\omega} x''_0 \right)$$

i.e.

$$\left(\Sigma_1^{x_0, x'_0} \circ [\hat{N}G]_1^{x'_0, x''_0} \right) \bullet^1 \left([MF]_1^{x_0, x'_0} \circ \left(([\alpha \bullet^0 F]_{x'_0} \circ \hat{\omega}_1^{x'_0, x''_0}) \bullet^1 ([\alpha \bullet^0 F]_1^{x'_0, x''_0} \circ \hat{\omega} x''_0) \right) \right)$$

By definition of 1-composition of 2-morphisms this is also

$$\left(\Sigma_1^{x_0, x'_0} \circ [\hat{N}G]_1^{x'_0, x''_0} \right) \bullet^1 \left([MF]_1^{x_0, x'_0} \circ [(\alpha \bullet^0 F) \bullet^1 \hat{\omega}]_1^{x'_0, x''_0} \right)$$

Symmetrical calculations can be made on the second 2-composite of (6.4), giving the composite

$$\begin{aligned} & \left(\Sigma_1^{x_0, x'_0} \circ [\hat{N}G]_1^{x'_0, x''_0} \right) \bullet^1 \left([MF]_1^{x_0, x'_0} \circ [(\alpha \bullet^0 F) \bullet^1 \hat{\omega}]_1^{x'_0, x''_0} \right) \\ & \quad \bullet^2 \\ & \left([\omega \bullet^1 (\beta \bullet^0 G)]_1^{x_0, x'_0} \circ [\hat{N}G]_1^{x'_0, x''_0} \right) \bullet^1 \left([MF]_1^{x_0, x'_0} \circ \Sigma_1^{x'_0, x''_0} \right) \end{aligned}$$

By composition coherence for Σ we get

$$(- \circ -) \bullet^0 \Sigma_1^{x_0, x''_0}$$

and by (6.3) again

$$(- \circ -) \bullet^0 \lambda_1^{x_0, x''_0} * \varepsilon_1^{Lx_0, \hat{L}x''_0}$$

Concluding, for every choice of three objects x_0, x'_0, x''_0 of \mathbb{X} the following three equations hold

$$\left\{ \begin{array}{l} \left((\lambda_1^{x_0, x'_0} \circ \hat{L}_1^{x'_0, x''_0}) \bullet^1 (L_1^{x_0, x'_0} \circ \lambda_1^{x'_0, x''_0}) \right) \bullet^0 Q_1^{Lx_0, \hat{L}x''_0} = (- \circ -) \bullet^0 \lambda_1^{x_0, x''_0} \bullet^0 Q_1^{x_0, x''_0} \\ \left((\lambda_1^{x_0, x'_0} \circ \hat{L}_1^{x'_0, x''_0}) \bullet^1 (L_1^{x_0, x'_0} \circ \lambda_1^{x'_0, x''_0}) \right) \bullet^0 P_1^{Lx_0, \hat{L}x''_0} = (- \circ -) \bullet^0 \lambda_1^{x_0, x''_0} \bullet^0 P_1^{x_0, x''_0} \\ \left((\lambda_1^{x_0, x'_0} \circ \hat{L}_1^{x'_0, x''_0}) \bullet^1 (L_1^{x_0, x'_0} \circ \lambda_1^{x'_0, x''_0}) \right) * \varepsilon_1^{Lx_0, \hat{L}x''_0} = (- \circ -) \bullet^0 \lambda_1^{x_0, x''_0} * \varepsilon_1^{Lx_0, \hat{L}x''_0} \end{array} \right.$$

Hence both $\left((\lambda_1^{x_0, x'_0} \circ \hat{L}_1^{x'_0, x''_0}) \bullet^1 (L_1^{x_0, x'_0} \circ \lambda_1^{x'_0, x''_0}) \right)$ and $(- \circ -) \bullet^0 \lambda_1^{x_0, x''_0}$ satisfy equations prescribed by universal property, hence by uniqueness they must be equal, and composition coherence is proved. \square

6.2 Ω and a second definition of π_1

We use the h -2pullback defined above to give an alternative description of the sesqui-functor $\pi_1^{(n)}$. A key observation is the analogy between the

$\text{hom}(n-1)$ -groupoid of a n -groupoid \mathbb{C} and the paths of a topological space.

Given a n -groupoid (n -category) \mathbb{C} and two objects c_0, c'_0 , we define $\mathbb{P}_{c_0, c'_0}(\mathbb{C})$ by means of the following h -pullback:

$$\begin{array}{ccc} \mathbb{P}_{c_0, c'_0}(\mathbb{C}) & \xrightarrow{!} & \mathbb{I}_{(n)} \\ \downarrow ! & \nearrow \varepsilon_{\mathbb{C}}^{c_0, c'_0} & \downarrow [c'_0] \\ \mathbb{I}_{(n)} & \xrightarrow{[c_0]} & \mathbb{C} \end{array} \quad (6.6)$$

This definition easily extends to morphisms. In fact for $F : \mathbb{C} \rightarrow \mathbb{D}$ one defines

$$\mathbb{P}_{c_0, c'_0}(F) : \mathbb{P}_{c_0, c'_0}(\mathbb{C}) \rightarrow \mathbb{P}_{c_0, c'_0}(\mathbb{D})$$

by means of the universal property of h -pullbacks yielding $\mathbb{P}_{c_0, c'_0}(\mathbb{D})$, for the four-tuple

$$\langle \mathbb{P}_{c_0, c'_0}(\mathbb{C}), !, \varepsilon_{\mathbb{C}}^{c_0, c'_0} \bullet^0 F \rangle$$

This makes $\mathbb{P}_{c_0, c'_0}(-)$ “somehow” functorial: in fact for $H : \mathbb{D} \rightarrow \mathbb{E}$,

$$\mathbb{P}_{c_0, c'_0}(F) \bullet^0 \mathbb{P}_{c_0, c'_0}(H) = \mathbb{P}_{c_0, c'_0}(F \bullet^0 H), \quad \mathbb{P}_{c_0, c'_0}(id_{\mathbb{C}}) = id_{\mathbb{P}_{c_0, c'_0}(\mathbb{C})}$$

Unfortunately this does not extend straightforward to 2-morphisms. In fact for a pair of parallel morphisms $F, G : \mathbb{C} \rightarrow \mathbb{D}$, $\mathbb{P}_{c_0, c'_0}(F)$ and $\mathbb{P}_{c_0, c'_0}(G)$ are no longer parallel, this making it difficult to extend $\mathbb{P}_{c_0, c'_0}(-)$ to natural n -transformations.

Indeed in applying the same argument as for defining $\mathbb{P}_{c_0, c'_0}(-)$ on morphisms, the corresponding diagram (shown below) suggests to consider the 0-composition of 2-morphisms

$$\varepsilon_{\mathbb{C}}^{c_0, c'_0} * \alpha : \varepsilon_{\mathbb{C}}^{c_0, c'_0} \setminus \alpha \Longrightarrow \varepsilon_{\mathbb{C}}^{c_0, c'_0} / \alpha$$

where as usual

$$\varepsilon_{\mathbb{C}}^{c_0, c'_0} \setminus \alpha = ([c_0] \bullet^0 \alpha) \bullet^1 (\varepsilon_{\mathbb{C}}^{c_0, c'_0} \bullet^0 G) = [\alpha_{c_0}] \bullet^1 (\varepsilon_{\mathbb{C}}^{c_0, c'_0} \bullet^0 G)$$

and

$$\varepsilon_{\mathbb{C}}^{c_0, c'_0} / \alpha = (\varepsilon_{\mathbb{C}}^{c_0, c'_0} \bullet^0 F) \bullet^1 ([c'_0] \bullet^0 \alpha) = (\varepsilon_{\mathbb{C}}^{c_0, c'_0} \bullet^0 F) \bullet^1 [\alpha_{c'_0}]$$

$$\begin{array}{ccc} \mathbb{P}_{c_0, c'_0}(\mathbb{C}) & \xrightarrow{!} & \mathbb{I}_{(n)} \\ \downarrow ! & \nearrow \varepsilon_{\mathbb{C}}^{c_0, c'_0} & \downarrow [Gc'_0] \\ \mathbb{I}_{(n)} & \xrightarrow{[Fc_0]} & \mathbb{D} \end{array} \quad \begin{array}{ccc} & \nearrow [c'_0] & \\ & \mathbb{C} & \\ & \searrow [c_0] & \\ & \mathbb{D} & \end{array} \quad \begin{array}{ccc} & \nearrow G & \\ & \alpha & \\ & \searrow F & \end{array}$$

Hence we can consider the four-ples

$$\langle \mathbb{P}_{c_0, c'_0}(\mathbb{C}), !, !, \varepsilon_{\mathbb{C}}^{c_0, c'_0} \setminus \alpha \rangle \quad \langle \mathbb{P}_{c_0, c'_0}(\mathbb{C}), !, !, \varepsilon_{\mathbb{C}}^{c_0, c'_0} / \alpha \rangle$$

$$\begin{array}{ccc} \mathbb{P}_{c_0, c'_0}(\mathbb{C}) & \xrightarrow{!} & \mathbb{I}_{(n)} \\ \downarrow ! & \nearrow \varepsilon_{\mathbb{C}}^{c_0, c'_0} \setminus \alpha & \downarrow [Gc'_0] \\ \mathbb{I}_{(n)} & \xrightarrow{[Fc_0]} & \mathbb{D} \end{array} \quad \begin{array}{ccc} \mathbb{P}_{c_0, c'_0}(\mathbb{C}) & \xrightarrow{!} & \mathbb{I}_{(n)} \\ \downarrow ! & \nearrow \varepsilon_{\mathbb{C}}^{c_0, c'_0} / \alpha & \downarrow [Gc'_0] \\ \mathbb{I}_{(n)} & \xrightarrow{[Fc_0]} & \mathbb{D} \end{array}$$

together with $id_! : ! \Rightarrow !$ (taken two times) and the 3-morphism $\varepsilon_{\mathbb{C}}^{c_0, c'_0} * \alpha$. Applying the universal property of h -2pullbacks we get a 2-morphism

$$\mathbb{P}_{c_0, c'_0}(\alpha) : \mathbb{P}_{[\alpha_{c_0}]} \circ \mathbb{P}_{c_0, c'_0}(G) \Rightarrow \mathbb{P}_{c_0, c'_0}(F) \circ \mathbb{P}_{[\alpha_{c'_0}]} : \mathbb{P}_{c_0, c'_0}(C) \rightarrow \mathbb{P}_{Fc_0, Gc'_0}(\mathbb{D})$$

such that

$$\begin{aligned} \mathbb{P}_{c_0, c'_0}(\alpha) \bullet^0 ! &= id_! \\ \mathbb{P}_{c_0, c'_0}(\alpha) \bullet^0 ! &= id_! \\ \mathbb{P}_{c_0, c'_0}(\alpha) * \varepsilon_{\mathbb{D}}^{Fc_0, Gc'_0} &= \varepsilon_{\mathbb{C}}^{c_0, c'_0} * \alpha \end{aligned} \quad (6.7)$$

Notice that we have denoted by $\mathbb{P}_{[\alpha_{c_0}]} \circ \mathbb{P}_{c_0, c'_0}(G)$ and $\mathbb{P}_{c_0, c'_0}(F) \circ \mathbb{P}_{[\alpha_{c'_0}]}$ the morphisms obtained by applying one-dimensional the universal property to $\varepsilon_{\mathbb{C}}^{c_0, c'_0} \setminus \alpha$ and $\varepsilon_{\mathbb{C}}^{c_0, c'_0} / \alpha$ respectively. Therefore the symbol \circ involved should be considered just a typographical suggestion. Indeed it can be shown that

it is a 0-composition of morphisms, but this would lead us far from the point. Furthermore it is inessential with respect to our purposes. For this reasons its further developing is left to the curious reader.

Purpose of the rest of the section is to give a characterization of π_1 as a consequence of the following

Theorem 6.5. *For every n -category \mathbb{C} , and every two objects c_0, c'_0 in \mathbb{C} , there exists a canonical isomorphism*

$$\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} : D(\mathbb{C}_1(c_0, c'_0)) \rightarrow \mathbb{P}_{c_0, c'_0}(\mathbb{C})$$

In the case of pointed n -groupoids, this gives a natural isomorphism with components

$$\mathfrak{S}_{\mathbb{C}}^{*,*} : D(\pi_1(\mathbb{C})) \rightarrow \Omega(\mathbb{C})$$

where $\Omega(\mathbb{C}) = \mathbb{P}_{*,*}(\mathbb{C})$

We start by making explicit h -pullback of (6.6), but first we need to be more precise on units.

Remark 6.6. Let \mathbb{C} be a n -category. For a fixed object c_0 of \mathbb{C} , let us consider the unit $(n-1)$ -functor given by the n -category structure of \mathbb{C} :

$$\mathbb{C}u^0(c_0) : \mathbb{I}_{(n-1)} \rightarrow \mathbb{C}_1(c_0, c_0)$$

We can make it explicit as a pair

$$\begin{aligned} [\mathbb{C}u^0(c_0)]_0 & : * \mapsto id(c_0) \in [\mathbb{C}_1(c_0, c'_0)]_0 \\ [\mathbb{C}u^0(c_0)]_1 & : \mathbb{I}_{(n-2)} \mapsto [\mathbb{C}_1(c_0, c'_0)]_1(id(c_0), id(c_0)) \end{aligned}$$

Now, by functoriality we get the interchange

$$[\mathbb{C}u^0(c_0)]_1 = \mathbb{C}u^1(id(c_0)) = \mathbb{C}_1(c_0, c_0)u^0(id(c_0))$$

and this allows the following explicit definition:

$$\mathbb{C}u^0(c_0) = \langle u^{(1)}(c_0), u^{(2)}(c_0), \dots, u^{(n)}(c_0) \rangle$$

where $u^{(k)}(c_0)$ is the identity k -cell over c_0 .

In the rest of this section, in order to simplify notation, the n -category $\mathbb{P}_{c_0, c'_0}(\mathbb{C})$ will be denoted by \mathbb{Q} .

Proposition 6.7. *Given the h -pullback of n -categories*

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{!} & \mathbb{I}_{(n)} \\ \downarrow ! & \nearrow \varepsilon & \downarrow [c'_0] \\ \mathbb{I}_{(n)} & \xrightarrow{[c_0]} & \mathbb{C} \end{array} \quad (6.8)$$

the $\text{hom}(n-k)$ -category

$$\mathbb{Q}_k \left(\left(u^{(k-1)}(*), c_{k-1} \xrightarrow{c_k} c'_{k-1}, u^{(k-1)}(*) \right), \left(u^{(k-1)}(*), c_{k-1} \xrightarrow{c'_k} c'_{k-1}, u^{(k-1)}(*) \right) \right)$$

is well defined and it is given by h -pullback over the diagram

$$\begin{array}{ccc} & \mathbb{I}_{(n-k)} & \\ & \downarrow [c'_k] & \\ \mathbb{I}_{(n-k)} & \xrightarrow{[c_k]} & \mathbb{C}_k(c_{k-1}, c'_{k-1}) \end{array}$$

Proof. By finite induction over k .

$$\boxed{k = 1}$$

We recall the definition of h -pullback:

\mathbb{Q}_0 is given by the limit in **Set**

$$\begin{array}{ccccc} & & \mathbb{Q}_0 & & \\ & \swarrow ! & \downarrow \varepsilon_0 & \searrow ! & \\ \{*\} = \mathbb{I}_0 & & [\mathbb{C}_1]_0 & & \mathbb{I}_0 = \{*\} \\ & \swarrow [c_0]_0 & \swarrow d \quad \searrow c & \swarrow [c'_0]_0 & \\ & \mathbb{C}_0 & & \mathbb{C}_0 & \end{array}$$

$\mathbb{Q}_1((*, c_1, *), (*, c'_1, *))$ is given by a h -pullback

$$\begin{array}{ccccc} \mathbb{Q}_1(\diamond, \diamond) & \xrightarrow{!} & \mathbb{I}_{(n-1)} & & \mathbb{Q}_1(\diamond, \diamond) \xrightarrow{!} \mathbb{I}_{(n-1)} \\ \downarrow ! & \nearrow \varepsilon_1^{\diamond, \diamond} & \downarrow [c'_0]_1 & & \downarrow [c'_1] \\ & & \mathbb{C}_1(c'_0, c'_0) = & & \\ & & \downarrow c_1 \circ - & & \\ \mathbb{I}_{(n-1)} & \xrightarrow{[c_0]_1} & \mathbb{C}_1(c_0, c_0) \xrightarrow{- \circ c'_1} \mathbb{C}_1(c_0, c'_0) & & \mathbb{I}_{(n-1)} \xrightarrow{[c_1]} \mathbb{C}_1(c_0, c'_0) \end{array}$$

with the symbol \diamond when the substitute is clear from the context.

$$\boxed{k > 1}$$

Induction hypothesis gives the following definition for

$$\mathbb{Q}_{k-1}\left((*, c_{k-2} \xrightarrow{c_{k-1}} c'_{k-2}, *), (*, c_{k-2} \xrightarrow{c'_{k-1}} c'_{k-2}, *)\right)$$

$$\begin{array}{ccc} \mathbb{Q}_{k-1}(\diamond, \diamond) & \xrightarrow{!} & {}_{(n-k+1)}\mathbb{I} \\ \downarrow ! & \nearrow \varepsilon_{k-1}^{\diamond, \diamond} & \downarrow [c'_{k-1}] \\ {}_{(n-k+1)}\mathbb{I} & \xrightarrow{[c_{k-1}]} & \mathbb{C}_{k-1}(c_{k-2}, c'_{k-2}) \end{array}$$

More explicitly, we get the following set-theoretical limit

$$\begin{array}{ccccc} & & [\mathbb{Q}_{k-1}(\diamond, \diamond)]_0 & & \\ & \swarrow ! & \downarrow [\varepsilon_{k-1}^{\diamond, \diamond}]_0 & \searrow ! & \\ \{*\} = [{}_{(n-k-1)}\mathbb{I}]_0 & & [\mathbb{C}_{k-1}(c_{k-2}, c'_{k-2})]_1 & & [{}_{(n-k-1)}\mathbb{I}]_0 = \{*\} \\ & \swarrow [c_0]_0 & \swarrow d & \searrow c & \swarrow [c'_0]_0 \\ & [\mathbb{C}_{k-1}(c_{k-2}, c'_{k-2})]_0 & & & [\mathbb{C}_{k-1}(c_{k-2}, c'_{k-2})]_0 \end{array}$$

i.e. the set $\{(*, c_{k-2} \xrightarrow{c_{k-1}} c'_{k-2}, *)\}$.

Hence $\mathbb{Q}_k((*, c_{k-1} \xrightarrow{c_k} c'_{k-1}, *), (*, c_{k-1} \xrightarrow{c'_k} c'_{k-1}, *))$ has (inductively)

well defined domain and codomain, namely $(*, c_{k-1} \xrightarrow{c_k} c'_{k-1}, *)$ and $(*, c_{k-1} \xrightarrow{c'_k} c'_{k-1}, *)$ are legitimate objects of a $\mathbb{Q}_{k-1}(\diamond, \diamond)$. By definition of h -pullback, we can spell it out:

$$\begin{array}{ccc} \mathbb{Q}_k((*, c_k, *), (*, c'_k, *)) & \xrightarrow{!} & [{}_{(n-k+1)}\mathbb{I}]_1 \\ \downarrow ! & \nearrow [\varepsilon_{k-1}^{\diamond, \diamond}]_1^{(*, c_k, *), (*, c'_k, *)} & \downarrow [c'_{k-1}]_1 \\ [{}_{(n-k+1)}\mathbb{I}]_1 & \xrightarrow{[c_{k-1}]_1} & [\mathbb{C}_{k-1}(c_{k-2}, c'_{k-2})]_1(c_{k-1}, c_{k-1}) \xrightarrow{- \circ c'_k} [\mathbb{C}_{k-1}(c_{k-2}, c'_{k-2})]_1(c_{k-1}', c'_{k-1}) \\ & & \downarrow c_k \circ - \\ & & [\mathbb{C}_{k-1}(c_{k-2}, c'_{k-2})]_1(c_{k-1}', c'_{k-1}) \end{array}$$

that may be rewritten

$$\begin{array}{ccc}
 \mathbb{Q}_k((*, c_k, *), (*, c'_k, *)) & \xrightarrow{!} & \mathbb{I}_{(n-k)} \\
 \downarrow ! & \nearrow \varepsilon_k^{(*, c_k, *), (*, c'_k, *)} & \downarrow [c'_k] \\
 \mathbb{I}_{(n-k)} & \xrightarrow{[c_k]} & \mathbb{C}_k(c_{k-1}, c'_{k-1})
 \end{array}$$

□

Proof of *Proposition 6.7* gives immediately the following

Corollary 6.8. *The 2-morphism ε is given explicitly by*

$$\varepsilon = < \varepsilon_0, [\varepsilon_1^-, -]_0, \dots, [\varepsilon_{n-1}^-, -]_0, >$$

where

$$\begin{aligned}
 [\varepsilon_k^{(*, c_{k-1}, *), (*, c'_{k-1}, *)}]_0 : \mathbb{Q}_k((*, c_{k-1}, *), (*, c'_{k-1}, *)) &\rightarrow \mathbb{C}_k(c_{k-1}, c'_{k-1}) \\
 (*, c_{k-1} \xrightarrow{c_k} c'_{k-1}, *) &\mapsto c_k
 \end{aligned}$$

Next Corollary states that h -pullbacks along two constants are n -discrete.

Corollary 6.9. *With notation as above,*

$$D(\pi_0(\mathbb{Q})) = \mathbb{Q}$$

Proof. It suffices to let $k = n$ in the above. $\mathbb{Q}_n((*, c_n, *), (*, c'_n, *))$ is given by the following pullback in **Set**:

$$\begin{array}{ccc}
 \mathbb{Q}_n((*, c_n, *), (*, c'_n, *)) & \xrightarrow{!} & \{*\} \\
 \downarrow ! & & \downarrow [c'_n] \\
 \{*\} & \xrightarrow{[c_n]} & \mathbb{C}_n(c_{n-1}, c'_{n-1})
 \end{array}$$

Hence, $\mathbb{Q}_n((*, c_n, *), (*, c'_n, *)) = \{*\}$ if $c_n = c'_n$, the *empty-set* otherwise. □

6.2.1 0-composition in $\mathbb{P}_{c_0, c'_0}(\mathbb{C})$

We have to describe 0-composition in $\mathbb{Q} = \mathbb{P}_{c_0, c'_0}(\mathbb{C})$, being all k -compositions (with $k > 0$) implicit in the inductive definition of \mathbb{Q} .

Let us start with functoriality of 2-morphism ε with respect to 0-composition. For every triple $(*, c_1, *)$, $(*, c'_1, *)$ and $(*, c''_1, *)$, diagram (3.5) may be written

$$\begin{array}{ccc}
& \mathbb{Q}_1((*, c_1, *), (*, c'_1, *)) \times \mathbb{Q}_1((*, c'_1, *), (*, c''_1, *)) & \\
\swarrow id \times [c'_1] & \xleftarrow{id \times \varepsilon_1^{(*, c'_0, *)}(*, c''_0, *)} & \searrow \varepsilon_1^{(*, c_0, *)}(*, c'_0, *) \times id \\
\mathbb{Q}_1((*, c_1, *), (*, c'_1, *)) \times \mathbb{C}_1(c_0, c'_0) & & \mathbb{C}_1(c_0, c'_0) \times \mathbb{Q}_1((*, c'_1, *), (*, c''_1, *)) \\
\searrow Pr_2 & & \swarrow Pr_1 \\
& \mathbb{C}_1(c_0, c'_0) &
\end{array}$$

(Note: The diagram above is a simplified representation of the complex commutative diagram in the image, which includes additional labels like $[c'_1] \times id$ and $id \times [c'_1]$ on the curved arrows.)

that reduces to

$$\begin{array}{ccc}
& \mathbb{Q}_1((*, c_1, *), (*, c'_1, *)) \times \mathbb{Q}_1((*, c'_1, *), (*, c''_1, *)) & \\
\swarrow [c'_1] & \xleftarrow{Pr_2 \circ \varepsilon_1^{(*, c'_0, *)}(*, c''_0, *)} & \searrow [c_1] \\
& \mathbb{C}_1(c_0, c'_0) &
\end{array}
\quad (6.9)$$

On the other side, the last term of equality (3.5) may be written

$$\begin{array}{ccc}
& \mathbb{Q}_1((*, c_1, *), (*, c'_1, *)) \times \mathbb{Q}_1((*, c'_1, *), (*, c''_1, *)) & \\
\downarrow \mathbb{Q}_0^0 & & \\
& \mathbb{Q}_1((*, c_1, *), (*, c''_1, *)) & \\
\swarrow [c'_1] & \xleftarrow{\varepsilon_1^{(*, c_0, *)}(*, c''_0, *)} & \searrow [c_1] \\
& \mathbb{C}_1(c_0, c'_0) &
\end{array}
\quad (6.10)$$

Comparing diagrams (6.9) and (6.10), the very definition of 0-compositions in h -pullbacks proves the following

Proposition 6.10. *Let $c_0, c'_0, c''_0 : c_0 \rightarrow c'_0$ be fixed in \mathbb{C} . Given*

$$c_k : c_1 = \Rightarrow c'_1, \quad c'_k : c_1 = \Rightarrow c'_1$$

with $1 < k \leq n$, the following equation holds:

$$(*, c_k, *)^{\mathbb{Q}} \circ^0 (*, c_k, *) = (*, c_k^{\mathbb{C}} \circ^1 c'_k, *)$$

Notation 6.11. *We use the notation $c_k : c_h = \Rightarrow c'_h$ with $h < k$, to mean that k -cell c_k has h -domain c_h and h -codomain c'_h , i.e. there exist cells $c_{k-1}, c'_{k-1} \dots c_h, c'_h$ such that*

$$c_k : c_{k-1} \rightarrow c'_{k-1} : c_{k-2} \rightarrow c'_{k-2} : \dots : c_{h+1} \rightarrow c'_{h+1} : c_h \rightarrow c'_h$$

Proof. (of Proposition 6.10) By definition of vertical composition of 2-morphisms and whiskering, diagram (6.9) gives

$$\begin{aligned} & \left[\left[\left(Pr1 \circ^0 \varepsilon_1^{(*, c_1, *)} \right) \circ^1 \left(Pr2 \circ^0 \varepsilon_1^{(*, c'_1, *)} \right) \right]_{k-1} \right]_0 ((*, c_k, *), (*, c'_k, *)) = \\ &= \left[\varepsilon_k^{(*, c_1, *)} \right]_0 ([Pr1]_{k-1} ((*, c_k, *), (*, c'_k, *)))^{\mathbb{C}} \circ^1 \dots \\ & \quad \dots \circ^1 \left[\varepsilon_k^{(*, c'_1, *)} \right]_0 ([Pr2]_{k-1} ((*, c_k, *), (*, c'_k, *))) \\ &= \left[\varepsilon_k^{(*, c_1, *)} \right]_0 ((*, c_k, *))^{\mathbb{C}} \circ^1 \left[\varepsilon_k^{(*, c'_1, *)} \right]_0 ((*, c'_k, *)) \\ &= c_k^{\mathbb{C}} \circ^1 c'_k \end{aligned}$$

where the last equality holds by *Corollary 6.8*. Next, diagram (6.10) gives

$$\begin{aligned} & \left[\left[-^{\mathbb{Q}} \circ^0 - \right] \circ^0 \varepsilon_1^{(*, c_1, *)} \right]_{k-1} \right]_0 ((*, c_k, *), (*, c'_k, *)) = \\ &= \left[\varepsilon_k^{(*, c_1, *)} \right]_0 ((*, c_k, *)^{\mathbb{Q}} \circ^0 (*, c'_k, *)) \\ &= \left[\varepsilon_k^{(*, c_1, *)} \right]_0 ((*, \bar{c}_k, *)) = \bar{c}_k \end{aligned}$$

Then, by comparison we get

$$(*, c_k, *)^{\mathbb{Q}} \circ^0 (*, c'_k, *) = (*, \bar{c}_k, *)$$

if, and only if,

$$\bar{c}_k = c_k^{\mathbb{C}} \circ^1 c'_k$$

and this completes the proof. \square

6.2.2 0-units in $\mathbb{P}_{c_0, c'_0}(\mathbb{C})$

We have to describe 0-units in $\mathbb{Q} = \mathbb{P}_{c_0, c'_0}(\mathbb{C})$, being all k -units (with $k > 0$) implicit in the inductive definition of \mathbb{Q} .

Let us start with functoriality of 2-morphism ε with respect to 0-units, for every $(*, c_1, *)$ in \mathbb{Q}_0 one can consider

$$\mathbb{Q}u^0((*, c_1, *)) : \mathbb{I}_{(n-1)} \longrightarrow \mathbb{Q}_1((*, c_1, *), (*, c_1, *))$$

Unit coherence (3.6) is then the equality

$$\begin{array}{ccc} \begin{array}{c} \mathbb{I}_{(n-1)} \\ \downarrow \mathbb{Q}u^0((*, c_1, *)) \\ \mathbb{Q}_1((*, c_1, *), (*, c_1, *)) \\ \begin{array}{ccc} [c_1] & \xleftarrow{\varepsilon_1^{(*, c_1, *), (*, c_1, *)}} & [c_1] \end{array} \\ \downarrow \\ \mathbb{C}_1(c_0, c'_0) \end{array} & = & \begin{array}{c} \mathbb{I}_{(n-1)} \\ \swarrow \quad \searrow \\ [c_1] \xleftarrow{id} [c_1] \\ \swarrow \quad \searrow \\ \mathbb{C}_1(c_0, c'_0) \end{array} \end{array}$$

This comparison, with the explicit description of ε given in *Corollary 6.8*, proves the following

Proposition 6.12. *Let $c_0 : c_0 \rightarrow c'_0$ be fixed in \mathbb{C} . For $1 < k \leq n$, the following equation holds*

$$\left[\mathbb{Q}u^0((*, c_1, *)) \right]_k = \left(*, \left[\mathbb{C}u^1(c_1) \right], * \right)$$

6.2.3 Comparison isomorphism \mathfrak{S}

What we are going to state provides an extremely powerful tool in developing the theory.

Lemma 6.13. *Let c_0, c'_0 be objects of an n -category \mathbb{C} . The assignment*

$$\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} = \mathfrak{S} : D(\mathbb{C}_1(c_0, c'_0)) \rightarrow \mathbb{P}_{c_0, c'_0}(\mathbb{C}) = \mathbb{Q}$$

given explicitly by

$$\mathfrak{S} = \langle \mathfrak{S}_0, \mathfrak{S}_1, \dots, \mathfrak{S}_n \rangle$$

with

$$\begin{aligned} \mathfrak{S}_{i-1} &:= c_i \mapsto (*, c_i, *), & i = 1, 2, \dots, n \\ \mathfrak{S}_n &:= \mathfrak{S}_{n-1} \end{aligned}$$

is an isomorphism of n -discrete n -categories.

Proof. By induction on n .

$\boxed{n = 1}$

The map $\mathfrak{S}(c_1) = (*, c_1, *)$ is trivially an isomorphism between discrete categories $D(\mathbb{C}_1(c_0, c'_0))$ and $\mathbb{P}_{c_0, c'_0}(\mathbb{C})$.

$\boxed{n > 1}$

Let us denote

$$\mathfrak{S} = \langle \mathfrak{S}_0, \{\mathfrak{S}_1, \dots, \mathfrak{S}_n\} \rangle .$$

In order for \mathfrak{S} to be an isomorphism of n -categories the following facts have to be checked:

1. \mathfrak{S}_0 is an isomorphism
2. for every pair $c_1, c'_1 : c_0 \rightarrow c'_0$,

$$\{\mathfrak{S}_1, \dots, \mathfrak{S}_n\}^{c_1, c'_1}$$

is an isomorphism of $(n - 1)$ -categories

3. above data satisfy usual functoriality axioms

1. Since $n > 1$, $[D(\mathbb{C}_1(c_0, c'_0))]_0 = [\mathbb{C}_1(c_0, c'_0)]_0$. Yet, by *Proposition 6.6* one has $[\mathbb{P}_{c_0, c'_0}(\mathbb{C})]_0 = \{*\} \times [\mathbb{C}_1(c_0, c'_0)]_0 \times \{*\}$. Hence the assignment $\mathfrak{S}_0(c_1) = (*, c_1, *)$ is clearly an isomorphism.

2. For any pair $c_1, c'_1 : c_0 \rightarrow c'_0$, induction hypothesis guaranties the existence of an isomorphism T^{c_1, c'_1} :

$$\begin{array}{ccc} [D(\mathbb{C}_1(c_0, c'_0))]_1(c_1, c'_1) & \xrightarrow{\quad\quad\quad} & [\mathbb{P}_{c_0, c'_0}(\mathbb{C})]_1((*, c_1, *), (*, c'_1, *)) \\ \parallel & & \parallel \\ D([\mathbb{C}_1(c_0, c'_0)]_1(c_1, c'_1)) & \xrightarrow{T^{c_1, c'_1}} & \mathbb{P}_{c_1, c'_1}(\mathbb{C}_1(c_0, c'_0)) \end{array}$$

defined by

$$\begin{aligned} T_{k-1}^{c_1, c'_1}(c_k) &= (*, c_k, *), & k = 2, \dots, n \\ T_n^{c_1, c'_1} &= T_{n-1}^{c_1, c'_1} \end{aligned}$$

Hence we let

$$\begin{aligned} \mathfrak{S}_{k-1} &= \coprod_{c_1, c'_1 \in \mathbb{C}_1(c_0, c'_0)} T_{k-1}^{c_1, c'_1}, & k = 2, \dots, n \\ \mathfrak{S}_n &= \mathfrak{S}_{n-1} \end{aligned}$$

so that the isomorphism T^{c_1, c'_1} is exactly $\{\mathfrak{S}_1, \dots, \mathfrak{S}_n\}^{c_1, c'_1}$.

3. We want to prove that $\langle \mathfrak{S}_0, T^{c_1, c'_1} \rangle$ is an (iso)morphism of n-categories, *i.e.* it satisfies usual coherence axioms.

• Let $c_1, c'_1, c''_1 : c_0 \rightarrow c'_0$ be given. Coherence w.r.t. composition amounts to the commutativity of the following diagram:

$$\begin{array}{ccc}
 [D(\mathbb{C}_1(c_0, c'_0))]_1(c_1, c'_1) \times [D(\mathbb{C}_1(c_0, c'_0))]_1(c'_1, c''_1) & \xrightarrow{D(\mathbb{C}_1(c_0, c'_0)) \circ^0} & [D(\mathbb{C}_1(c_0, c'_0))]_1(c_1, c''_1) \\
 \parallel & (i) & \parallel \\
 D(\mathbb{C}_2(c_1, c'_1) \times \mathbb{C}_2(c'_1, c''_1)) & \xrightarrow{D(\mathbb{C}_0^1)} & D(\mathbb{C}_2(c_1, c''_1)) \\
 \parallel & & \downarrow T^{c_1, c''_1} \\
 D(\mathbb{C}_2(c_1, c'_1)) \times D(\mathbb{C}_2(c'_1, c''_1)) & (ii) & \\
 \downarrow T^{c_1, c'_1} \times T^{c'_1, c''_1} & & \\
 \mathbb{Q}_1((*, c_1, *), (*, c'_1, *)) \times \mathbb{Q}_1((*, c'_1, *), (*, c''_1, *)) & \xrightarrow{\mathbb{Q}_0} & \mathbb{Q}_1((*, c_1, *), (*, c''_1, *))
 \end{array}$$

Here (i) commutes by definition, while (ii) commutes *point-wise*. In fact, for any $k = 2, \dots, n$ and for $c_k : c_1 \Rightarrow c'_1$ and $c'_k : c'_1 \Rightarrow c''_1$ Proposition 6.10 gives

$$\begin{array}{ccc}
 (c_k, c'_k) & \xrightarrow{D(\mathbb{C}_0^1)} & c_k \circ^1 c'_k \\
 \downarrow T^{c_1, c'_1} \times T^{c'_1, c''_1} & & \downarrow T^{c_1, c''_1} \\
 ((*, c_k, *), (*, c'_k, *)) & \xrightarrow{\mathbb{C}_0} & (*, c_k \circ^1 c'_k, *)
 \end{array}$$

• Let $c_1 : c_0 \rightarrow c : 0'$ be given. Coherence w.r.t. units amounts to the commutativity of the following diagram:

$$\begin{array}{ccc}
 \mathbb{I}_{(n-1)} & \xrightarrow{D(\mathbb{C}_1(c_0, c'_0)) u^0(c_1)} & [D(\mathbb{C}_1(c_0, c'_0))] \\
 & \searrow c_{u^1(c_1)} & \parallel \\
 & & D(\mathbb{C}_2(c_1, c_1)) \\
 & \searrow \mathbb{Q}_{u^0}(*, c_1, *) & \downarrow T^{c_1} \\
 & & \mathbb{Q}_1((*, c_1, *), (*, c_1, *))
 \end{array}$$

Upper triangle commutes by definition, lower triangle commutes *point-wise*. In fact, for any $k = 2, \dots, n$ *Proposition 6.12* gives

$$\begin{array}{ccc}
 * & \xrightarrow{[\mathbb{C}u^1(c_1)]_k} & [\mathbb{C}u^1(c_1)]_k(*) \\
 & \searrow [\mathbb{Q}u^0(*, c_1, *)]_k & \downarrow T^{c_1, c_1} \\
 & & (*, [\mathbb{C}u^1(c_1)]_k(*), *)
 \end{array}$$

Finally, in the proof we did not explicit the level of \mathfrak{S}_n , as n-categories considered are n -discrete and sesqui-functor D is a full inclusion. \square

6.2.4 Back to the Theorem

Now that we have developed the machinery, we are able to prove the main theorem of the section.

Proof of Theorem 6.5. The previous Lemma guaranties precisely the existence of a canonical isomorphism of n-categories

$$\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} : D(\mathbb{C}_1(c_0, c'_0)) \rightarrow \mathbb{P}_{c_0, c'_0}(\mathbb{C}) = \mathbb{Q}$$

for any pair of objects c_0, c'_0 . Further, for a n-functor $F : \mathbb{C} \rightarrow \mathbb{D}$ we get a (c_0, c'_0) -indexed family of commutative squares:

$$\begin{array}{ccc}
 D(\mathbb{C}_1(c_0, c'_0)) & \xrightarrow{D(F_1^{c_0, c'_0})} & D(\mathbb{D}_1(Fc_0, Fc'_0)) \\
 \mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} \downarrow & & \downarrow \mathfrak{S}_{\mathbb{D}}^{Fc_0, Fc'_0} \\
 \mathbb{P}_{c_0, c'_0}(\mathbb{C}) & \xrightarrow{\mathbb{P}_{c_0, c'_0}(F)} & \mathbb{P}_{Fc_0, Fc'_0}(\mathbb{D})
 \end{array} \tag{6.11}$$

We prove this by induction.

For $n = 1$ it is just a diagram of discrete categories. It suffices to verify commutativity on objects. To this end, let us choose a $c_1 : c_0 \rightarrow c'_0$. Equations below complete the case:

$$\begin{aligned}
 \mathfrak{S}_{\mathbb{D}}(DF(c_1)) &= \mathfrak{S}_{\mathbb{D}}(Fc_1) \\
 &= (*, Fc_1, *) \\
 \mathbb{P}F(\mathfrak{S}_{\mathbb{C}}(c_1)) &= \mathbb{P}F(*, c_1, *) \\
 &= (*, Fc_1, *).
 \end{aligned}$$

Hence let us consider a generic $n > 1$. First we have to show that diagram (6.11) commutes on objects, but this amounts exactly to what we have just shown for $n = 1$.

Thus we fix $c_1, c'_1 : c_0 \rightarrow c'_0$ and consider homs:

$$\begin{array}{ccc} [D(\mathbb{C}_1(c_0, c'_0))]_1(c_1, c'_1) & \xrightarrow{[D(F_1^{c_0, c'_0})]_1^{c_1, c'_1}} & [D(\mathbb{D}_1(Fc_0, Fc'_0))]_1(Fc_1, Fc'_1) \\ \downarrow [\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0}]_1^{c_1, c'_1} & & \downarrow [\mathfrak{S}_{\mathbb{D}}^{Fc_0, Fc'_0}]_1^{Fc_1, Fc'_1} \\ [\mathbb{P}_{c_0, c'_0}(\mathbb{C})]_1((*, c_1, *), (*, c'_1, *)) & \xrightarrow{[\mathbb{P}_{c_0, c'_0}(F)]_1^{\diamond, \diamond}} & [\mathbb{P}_{Fc_0, Fc'_0}(\mathbb{D})]_1((*, Fc_1, *), (*, Fc'_1, *)) \end{array}$$

The definition of discretizer functor D allows us to re-formulate the diagram as follows:

$$\begin{array}{ccc} D([\mathbb{C}_1(c_0, c'_0)]_1(c_1, c'_1)) & \xrightarrow{D([F_1^{c_0, c'_0}]_1^{c_1, c'_1})} & D([\mathbb{D}_1(c_0, c'_0)]_1(Fc_1, Fc'_1)) \\ \downarrow [\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0}]_1^{c_1, c'_1} & & \downarrow [\mathfrak{S}_{\mathbb{D}}^{Fc_0, Fc'_0}]_1^{Fc_1, Fc'_1} \\ [\mathbb{P}_{c_0, c'_0}(\mathbb{C})]_1((*, c_1, *), (*, c'_1, *)) & \xrightarrow{[\mathbb{P}_{c_0, c'_0}(F)]_1^{\diamond, \diamond}} & [\mathbb{P}_{Fc_0, Fc'_0}(\mathbb{D})]_1((*, Fc_1, *), (*, Fc'_1, *)) \end{array}$$

and the previous discussion turns it in

$$\begin{array}{ccc} D([\mathbb{C}_1(c_0, c'_0)]_1(c_1, c'_1)) & \xrightarrow{D([F_1^{c_0, c'_0}]_1^{c_1, c'_1})} & D([\mathbb{D}_1(c_0, c'_0)]_1(Fc_1, Fc'_1)) \\ \downarrow T_{\mathbb{C}}^{c_1, c'_1} & & \downarrow T_{\mathbb{D}}^{Fc_1, Fc'_1} \\ \mathbb{P}_{c_1, c'_1}(\mathbb{C}_1(c_0, c'_0)) & \xrightarrow{\mathbb{P}_{c_1, c'_1}(F_1^{c_0, c'_0})} & \mathbb{P}_{Fc_1, Fc'_1}(\mathbb{D}_1(Fc_0, Fc'_0)) \end{array}$$

Now, as T 's are just \mathfrak{S} 's given for $n-1$, *i.e.*

$$T_{\mathbb{C}}^{c_1, c'_1} = \mathfrak{S}_{\mathbb{C}_1(c_0, c'_0)}^{c_1, c'_1} \quad \text{and} \quad T_{\mathbb{D}}^{Fc_1, Fc'_1} = \mathfrak{S}_{\mathbb{D}_1(Fc_0, Fc'_0)}^{Fc_1, Fc'_1}$$

the last diagram commutes by induction hypothesis.

It is clear that all this restricts to n -groupoids. Moreover, in pointed case we obtain a 2-contravariant natural isomorphism of sesqui-functors, *i.e.* a strict natural transformation of sesqui-functors that reverses the direction of 2-morphisms and in which the assignments on objects are isomorphisms:

$$\begin{array}{ccc} & \xrightarrow{\pi_1} (n-1)\mathbf{Gpd}_* & \xrightarrow{D} \\ n\mathbf{Gpd}_* & & n\mathbf{Gpd}_* \\ & \Downarrow \mathfrak{S} & \\ & \Omega & \end{array}$$

Indeed in $n\mathbf{Gpd}_*$ (i.e. in $n\mathbf{Gpd}$ with $c_0 = * = c'_0$), for 2-morphism $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$, we can express the (strict) naturality condition

$$\begin{array}{ccc}
 & D(G_1^{*,*}) & \\
 & \curvearrowright & \\
 D(\mathbb{C}_1(*, *)) & \xrightarrow{D(\alpha_1^{*,*})} & D(\mathbb{D}_1(*, *)) \\
 & \curvearrowleft & \\
 & D(F_1^{*,*}) & \\
 \downarrow \mathfrak{S}_{\mathbb{C}}^{*,*} & & \downarrow \mathfrak{S}_{\mathbb{D}}^{*,*} \\
 \mathbb{P}_{*,*}(\mathbb{C}) & \xrightarrow{\mathbb{P}_{*,*}(G)} & \mathbb{P}_{*,*}(\mathbb{D}) \\
 & \curvearrowleft & \\
 & \mathbb{P}_{*,*}(F) & \\
 & \curvearrowright &
 \end{array}
 \quad
 \begin{array}{l}
 D(\alpha_1^{*,*}) \bullet^0 \mathfrak{S}_{\mathbb{D}}^{*,*} \\
 = \\
 \mathfrak{S}_{\mathbb{C}}^{*,*} \bullet^0 \mathbb{P}_{*,*}(\alpha)
 \end{array}$$

The proof that this condition indeed holds is a corollary to the following *Lemma*, that therefore concludes the proof. \square

Lemma 6.14. *Given the 2-morphism of n -groupoids $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$, then the following equation holds.*

$$\begin{array}{ccc}
 & D(\alpha_{c_0} \circ G_1^{c_0, c'_0}) & \\
 & \parallel & \\
 D(\mathbb{C}_1(c_0, c'_0)) & \xrightarrow{D(\alpha_1^{c_0, c'_0})} & D(\mathbb{D}_1(Fc_0, Gc'_0)) \\
 & \parallel & \\
 & D(F_1^{c_0, c'_0} \circ \alpha_{c'_0}) & \\
 \downarrow \mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} & & \downarrow \mathfrak{S}_{\mathbb{D}}^{Fc_0, Gc'_0} \\
 \mathbb{P}_{c_0, c'_0}(\mathbb{C}) & \xrightarrow{\mathbb{P}_{[\alpha_{c_0}]} \circ \mathbb{P}_{c_0, c'_0}(G)} & \mathbb{P}_{Fc_0, Gc'_0}(\mathbb{D}) \\
 & \parallel & \\
 & \mathbb{P}_{c_0, c'_0}(\alpha) & \\
 & \downarrow & \\
 & \mathbb{P}_{c_0, c'_0}(F) \circ \mathbb{P}_{[\alpha_{c'_0}]} &
 \end{array}
 \quad
 \begin{array}{l}
 D(\alpha_1^{c_0, c'_0}) \bullet^0 \mathfrak{S}_{\mathbb{D}}^{Fc_0, Gc'_0} \\
 = \\
 \mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} \bullet^0 \mathbb{P}_{c_0, c'_0}(\alpha)
 \end{array}$$

Proof. We will prove the *Lemma* by means of the universal property of h -2pullback defining $\mathbb{P}_{Fc_0, Gc'_0}(\mathbb{D})$. To this end let us first consider the following quite trivial chain of equalities (taken two times):

$$D(\alpha_1^{c_0, c'_0}) \bullet^0 \mathfrak{S}_{\mathbb{D}}^{Fc_0, Gc'_0} \bullet^0 ! = D(\alpha_1^{c_0, c'_0}) \bullet^0 ! = [id!] = \mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} \bullet^0 [id!] = \mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} \bullet^0 \mathbb{P}_{c_0, c'_0}(\alpha) \bullet^0 !$$

Less trivially we want to prove

$$(D(\alpha_1^{c_0, c'_0}) \bullet^0 \mathfrak{S}_{\mathbb{D}}^{Fc_0, Gc'_0}) * \varepsilon_{\mathbb{D}}^{Fc_0, Gc'_0} = (\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} \bullet^0 \mathbb{P}_{c_0, c'_0}(\alpha)) * \varepsilon_{\mathbb{D}}^{Fc_0, Gc'_0}$$

By equation (6.7) this can be rewritten

$$D(\alpha_1^{c_0, c'_0}) * (\mathfrak{S}_{\mathbb{D}}^{Fc_0, Gc'_0} \bullet^0 \varepsilon_{\mathbb{D}}^{Fc_0, Gc'_0}) = (\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} \bullet^0 \varepsilon_{\mathbb{C}}^{c_0, c'_0}) * \alpha$$

Let us prove directly the equality on objects. To this end let us fix an arbitrary “object” c_1 of $D(\mathbb{C}_1^{c_0, c'_0})$. Then applying definitions we get

$$\begin{aligned}
& \left[D(\alpha_1^{c_0, c'_0}) * \left(\mathfrak{S}_{\mathbb{D}}^{Fc_0, Gc'_0} \bullet^0_{\varepsilon_{\mathbb{D}}} \right) \right]_0 (c_1) = \\
&= \left[\mathfrak{S}_{\mathbb{D}}^{Fc_0, Gc'_0} \bullet^0_{\varepsilon_{\mathbb{D}}} \right]_1 \left(\left[D(\alpha_1^{c_0, c'_0}) \right]_0 (c_1) \right) \\
&= \left[\mathfrak{S}_{\mathbb{D}}^{Fc_0, Gc'_0} \bullet^0_{\varepsilon_{\mathbb{D}}} \right]_1 \left(\alpha_{c_0} \circ Gc_1 \left(\begin{array}{c} \xrightarrow{Fc_0} \\ \xrightarrow{\alpha_{c_1}} \\ \xrightarrow{Gc'_0} \end{array} Fc_1 \circ \alpha_{c'_0} \right) \right) \\
&= \left[\varepsilon_{\mathbb{D}}^{Fc_0, Gc'_0} \right]_1 \left(\mathfrak{S}_{\mathbb{D}}^{Fc_0, Gc'_0}(\alpha_{c_1}) \right) \\
&= \left[\varepsilon_{\mathbb{D}}^{Fc_0, Gc'_0} \right]_1 \left((*, \alpha_{c_1}, *) \right) \\
&= \alpha_{c_1} \\
&= \alpha_1 \left(\left[\varepsilon_{\mathbb{C}}^{c_0, c'_0} \right]_0 \left((*, c_1, *) \right) \right) \\
&= \alpha_1 \left(\left[\varepsilon_{\mathbb{C}}^{c_0, c'_0} \right]_0 \left(\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0}(c_1) \right) \right) \\
&= \alpha_1 \left(\left[\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} \bullet^0_{\varepsilon_{\mathbb{C}}} \right]_0 (c_1) \right) \\
&= \left[(\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} \bullet^0_{\varepsilon_{\mathbb{C}}}) * \alpha \right]_0 (c_1)
\end{aligned}$$

On homs we will proceed by induction. Hence let us fix arbitrary “objects” c_1, c'_1 of $D(\mathbb{C}_1^{c_0, c'_0})$. Then

$$\begin{aligned}
& \left[D(\alpha_1^{c_0, c'_0}) * \left(\mathfrak{S}_{\mathbb{D}}^{Fc_0, Gc'_0} \bullet^0 \varepsilon_{\mathbb{D}}^{Fc_0, Gc'_0} \right) \right]_1^{c_1, c'_1} = \\
& \stackrel{(i)}{=} \left[D(\alpha_1^{c_0, c'_0}) \right]_1^{c_1, c'_1} * \left[\mathfrak{S}_{\mathbb{D}}^{Fc_0, Gc'_0} \bullet^0 \varepsilon_{\mathbb{D}}^{Fc_0, Gc'_0} \right]_1^{\alpha_{c_0} \circ Gc_1, Fc'_1 \circ \alpha_{c'_0}} \\
& \stackrel{(ii)}{=} \left[D(\alpha_1^{c_0, c'_0}) \right]_1^{c_1, c'_1} * \left(\left[\mathfrak{S}_{\mathbb{D}}^{Fc_0, Gc'_0} \right]_1^{\alpha_{c_0} \circ Gc_1, Fc'_1 \circ \alpha_{c'_0}} \bullet^0 \left[\varepsilon_{\mathbb{D}}^{Fc_0, Gc'_0} \right]_1^{(*, \alpha_{c_0} \circ Gc_1, *), (*, Fc'_1 \circ \alpha_{c'_0}, *)} \right) \\
& \stackrel{(iii)}{=} \left[D(\alpha_1^{c_0, c'_0}) \right]_1^{c_1, c'_1} * \left(\mathfrak{S}_{\mathbb{D}_1}^{\alpha_{c_0} \circ Gc_1, Fc'_1 \circ \alpha_{c'_0}} \bullet^0 \varepsilon_{\mathbb{D}_1}^{(*, \alpha_{c_0} \circ Gc_1, *), (*, Fc'_1 \circ \alpha_{c'_0}, *)} \right) \\
& \stackrel{(iv)}{=} D \left(\left[\alpha_1^{c_0, c'_0} \right]_1^{c_1, c'_1} \right) * \left(\mathfrak{S}_{\mathbb{D}_1}^{\alpha_{c_0} \circ Gc_1, Fc'_1 \circ \alpha_{c'_0}} \bullet^0 \varepsilon_{\mathbb{D}_1}^{(*, \alpha_{c_0} \circ Gc_1, *), (*, Fc'_1 \circ \alpha_{c'_0}, *)} \right) \\
& \stackrel{(v)}{=} \left(\mathfrak{S}_{\mathbb{C}_1^{c_0, c'_0}}^{c_1, c'_1} \bullet^0 \varepsilon_{\mathbb{C}_1^{c_0, c'_0}}^{(*, c_1, *), (*, c'_1, *)} \right) * \alpha_1^{c_0, c'_0} \\
& \stackrel{(vi)}{=} \left(\left[\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} \right]_1^{c_1, c'_1} \bullet^0 \left[\varepsilon_{\mathbb{C}}^{c_0, c'_0} \right]_1^{(*, c_1, *), (*, c'_1, *)} \right) * \alpha_1^{c_0, c'_0} \\
& \stackrel{(vii)}{=} \left[\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} \bullet^0 \varepsilon_{\mathbb{C}}^{c_0, c'_0} \right]_1^{c_1, c'_1} * \alpha_1^{c_0, c'_0} \\
& \stackrel{(viii)}{=} \left[(\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0} \bullet^0 \varepsilon_{\mathbb{C}}^{c_0, c'_0}) * \alpha \right]_1^{c_1, c'_1}
\end{aligned}$$

where (i) and (viii) hold by definition of $*$ -composition, (ii) and (vii) by definition of 0-whiskering for 2-morphisms, (iii) and (vi) by the inductive definition of \mathfrak{S} and ε , (iv) by definition of discretizer and finally (v) by induction hypothesis.

Uniqueness provided by the universal property completes the proof. \square

At last the promised alternative description of π_1 .

Corollary 6.15. *Let \mathbb{C} be a pointed n -groupoid. Then there exists a natural isomorphism of sesqui-functors with components*

$$\pi_0^{(n)}(\mathfrak{S}_{\mathbb{C}}^{*,*}) : \pi_1^{(n)}(\mathbb{C}) \rightarrow \pi_0^{(n)}(\Omega(\mathbb{C}))$$

Proof. Since $\mathfrak{S}_{\mathbb{C}}^{*,*}$ is n -discrete, $\pi_0^{(n)}(\mathfrak{S}_{\mathbb{C}}^{*,*})$ is still an isomorphism. \square

Remark 6.16. From now on, as a consequence of *Corollary 6.15* we will often identify the sesqui-functors $\pi_1(-)$ and $\pi_0(\Omega(-))$.

6.2.5 Final remark on \mathfrak{S}

As the reader may guess, \mathfrak{S} is of a richer nature than we have shown in previous sections.

As a matter of fact we have developed the theory as far as our purposes require. Nevertheless we urge to give a hint of the big picture behind.

It is possible to define a sesqui-functor

$$\tilde{\Sigma}^{(n)} = \tilde{\Sigma} : n\mathbf{Cat} \rightarrow (n+1)\mathbf{Cat}$$

that assigns to a n -category \mathbb{C} the $(n+1)$ -category $\tilde{\Sigma}(\mathbb{C})$ with two distinguished objects $*_0$ and $*_1$, and the following hom- n -categories

- $[\tilde{\Sigma}(\mathbb{C})]_1(*_0, *_0) = \mathbb{I}_{(n)}$
- $[\tilde{\Sigma}(\mathbb{C})]_1(*_0, *_1) = \mathbb{C}$
- $[\tilde{\Sigma}(\mathbb{C})]_1(*_1, *_0) = \emptyset$
- $[\tilde{\Sigma}(\mathbb{C})]_1(*_1, *_1) = \mathbb{I}_{(n)}$

with trivial compositions and obvious units.

Then our comparison gives natural isomorphism

$$\mathfrak{S}_{\tilde{\Sigma}(\mathbb{C})}^{*0, *1} : D(\mathbb{C}) \rightarrow \mathbb{P}_{*0, *1}(\tilde{\Sigma}(\mathbb{C})).$$

If we restrict to equivalence sesqui-categories \mathbf{Cat}_{eq} , this allows to represent any n -category as a $(\pi_0$ of a) specified h -pullback. Moreover our $\mathfrak{S}_{\mathbb{C}}^{c_0, c'_0}$ are indeed hom- n -categories

$$\left[\mathbb{P}_{*0, *1}(\tilde{\Sigma}(\mathbb{C})) \right]_1((*, c_0, *), (*, c'_0, *))$$

(this justifying the notation adopted) of this representation, and it is easy to check that compositions and units are provided by the universal property of h -pullbacks.

All this suggests to further develop the theory in the direction of homotopy theory. In fact the construction developed for $\mathbb{P}_{-, -}(-)$ is indeed obtained by the more general product preserving *path-functor*

$$\mathbb{P} : n\mathbf{Cat} \rightarrow n\mathbf{Cat}$$

defined by the h -pullback

$$\begin{array}{ccc} \mathbb{P}(\mathbb{C}) & \xrightarrow{C} & \mathbb{C} \\ D \downarrow & \nearrow \varepsilon_C & \parallel \\ \mathbb{C} & \xlongequal{\quad} & \mathbb{C} \end{array}$$

This is a universal 2-morphisms representor, in that it represents every 2-cell $\alpha : F \Rightarrow G : \mathbb{B} \rightarrow \mathbb{C}$ as a functor $\tilde{\alpha} : \mathbb{B} \rightarrow \mathbb{P}(\mathbb{C})$ such that $D(\tilde{\alpha}) = F$, $C(\tilde{\alpha}) = G$ and $\varepsilon_{\mathbb{C}}(\tilde{\alpha}) = \alpha$ (compare with universal property of h -pullback). Moreover the existence of a path-functor permits to obtain all h -pullbacks as ordinary limits. In fact it is easy to show that the h -pullback over

$$\mathbb{A} \xrightarrow{F} \mathbb{B} \xleftarrow{G} \mathbb{C}$$

is indeed the usual categorical limit over

$$\mathbb{A} \xrightarrow{F} \mathbb{B} \xleftarrow{D} \mathbb{P}(\mathbb{B}) \xrightarrow{C} \mathbb{B} \xleftarrow{G} \mathbb{C}$$

All this can be made absolutely precise and algebraic by considering the notion of *cubical comonad* and related structures on it [Gra97].

6.3 Monoidal structure on $\Omega(\mathbb{C})$

Let \mathbb{C} be a pointed n -groupoid. In applying the loop sesqui-functor Ω to \mathbb{C} one notices that the new structure has no memory of the 0-composition in \mathbb{C} . In fact, every cell of $\Omega(\mathbb{C})$, *i.e.* every cell of \mathbb{C} with the object “ $*$ ” as its 0-domain and 0-codomain is 0-composable. Hence we can recover the forgotten structure, in order to get a *many sorted* strict monoidal structure

$$(\Omega(\mathbb{C}), \otimes, I)$$

with $\otimes = \circ^0$, and $I = (*, 1_*, *)$. Moreover cells are weakly invertible w.r.t. this structure, thus giving a *group like* structure to the n -groupoid $\Omega(\mathbb{C})$.

Remark 6.17. This is indeed the same as considering $\Omega(\mathbb{C})$ as a $(n+1)$ -groupoid with one object.

n -Functoriality axioms prove the following

Lemma 6.18. *The following two statements hold for n -groupoids, $n > 0$.*

1. *Let n -functor $F : \mathbb{C} \rightarrow \mathbb{D}$ be given. Then $\Omega(F)$ is a strict monoidal n -functor.*
2. *Let natural n -transformation $\alpha : F \Rightarrow G : \mathbb{C} \rightarrow \mathbb{D}$ be given. Then $\Omega(F)$ is a strict monoidal natural n -transformation.*

Furthermore if we apply Ω once more, we get another monoidal structure on the n -discrete $(n-1)$ -discrete n -groupoid $\Omega(\Omega(\mathbb{C}))$. This new structure corresponds to 0-composition of $\Omega(\mathbb{C})$, *i.e.* 1-composition of \mathbb{C} . Functoriality of units and compositions allow us to apply an Heckmann-Hilton like argument, this showing that compositions coincide and are indeed commutative.

In terms of monoidal structures, we can resume the previous discussion:

$$(\Omega(\Omega(\mathbb{C})), \otimes, I)$$

is a commutative strict monoidal structure on the n -groupoid $\Omega(\Omega(\mathbb{C}))$.

Notice that monoidal structure is automatically preserved by π_0 , hence all this can be said for sesqui-functor π_1 :

Proposition 6.19. *Let \mathbb{C} be a n -groupoid. Then the $(n-1)$ -groupoid $\pi_1(\mathbb{C})$ is naturally endowed with weakly invertible strict monoidal structure. This structure is commutative when we consider $(\pi_1)^2(\mathbb{C})$.*

6.4 Ω and π_1 preserve exactness

In the following paragraphs we will show that, given a three-term exact sequence in $n\mathbf{Gpd}$, the sesqui-functor $\Omega^{(n)}$ produces a three-term exact sequence in $n\mathbf{Gpd}$. As a consequence, we obtain a similar result for $\pi_1^{(n)}$

Lemma 6.20. *Sesqui-functor $\Omega^{(n)}$ preserves h -pullbacks.*

Proof. We consider a slightly more general setting, in order to get the proof of the statement as a consequence. Notice that we will omit the superscripts (n) .

Let us consider a h -pullback

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{Q} & \mathbb{C} \\ P \downarrow & \nearrow \phi & \downarrow G \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} \end{array}$$

in $n\mathbf{Gpd}$. Moreover let us fix objects $\alpha \in \mathbb{A}_0$ and $\gamma \in \mathbb{C}_0$ such that $F(\alpha) = \beta = G(\gamma)$. Next let us apply the universal property of h -2pullback to get $\mathbb{P}(\phi)$ as in the diagram

$$\begin{array}{ccc} \mathbb{P}_{q,q}(\mathbb{Q}) & \xrightarrow{\mathbb{P}(P)} & \mathbb{P}_{\alpha,\alpha}(\mathbb{A}) \\ \mathbb{P}(Q) \downarrow & \nearrow \mathbb{P}(\phi) & \downarrow \mathbb{P}(F) \\ \mathbb{P}_{\gamma,\gamma}(\mathbb{C}) & \xrightarrow{\mathbb{P}(G)} & \mathbb{P}_{\beta,\beta}(\mathbb{B}) \end{array}$$

where $q = (\alpha, 1_\beta, \gamma)$. Further let us consider the diagram

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{M} & \mathbb{P}_{\alpha,\alpha}(\mathbb{A}) \\ N \downarrow & \nearrow \omega & \downarrow \mathbb{P}(F) \\ \mathbb{P}_{\gamma,\gamma}(\mathbb{C}) & \xrightarrow{\mathbb{P}(G)} & \mathbb{P}_{\beta,\beta}(\mathbb{B}) \end{array}$$

that by *Theorem 6.5* can be re-drawn

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{M} & D(\mathbb{A}_1(\alpha, \alpha)) \\ N \downarrow & \nearrow \omega & \downarrow D(F_1^{\alpha, \alpha}) \\ D(\mathbb{C}_1(\gamma, \gamma)) & \xrightarrow{D(G_1^{\gamma, \gamma})} & D(\mathbb{B}_1(\beta, \beta)) \end{array}$$

Applying π_0 one gets

$$\begin{array}{ccc} \pi_0(\mathbb{X}) & \xrightarrow{\pi_0(M)} & \mathbb{A}_1(\alpha, \alpha) \\ \pi_0(N) \downarrow & \nearrow \pi_0(\omega) & \downarrow F_1^{\alpha, \alpha} \\ \mathbb{C}_1(\gamma, \gamma) & \xrightarrow{G_1^{\gamma, \gamma}} & \mathbb{B}_1(\beta, \beta) \end{array}$$

Since $\mathbb{Q}_1(q, q)$ is defined as a h -pullback (see Section 3.6), the universal property yields a unique

$$L : \pi_0(\mathbb{X}) \rightarrow \mathbb{Q}_1(q, q)$$

such that

$$(i) L \bullet^0 P_1^{q, q} = \pi_0(M) \quad (ii) L \bullet^0 Q_1^{q, q} = \pi_0(N) \quad (iii) L \bullet^0 \phi_1^{q, q} = \pi_0(\omega)$$

which in turn implies that *there exists a unique*

$$\mathbb{X} \xrightarrow{\eta_{\mathbb{X}}} D\pi_0(\mathbb{X}) \xrightarrow{DL} \mathbb{D}(\mathbb{Q}_1(q, q)) = \mathbb{P}_{q, q}(\mathbb{Q})$$

such that

$$\begin{aligned} \eta_{\mathbb{X}} \bullet^0 DL \bullet^0 \mathbb{P}(P) &= \eta_{\mathbb{X}} \bullet^0 DL \bullet^0 DP_1^{q, q} \\ &= \eta_{\mathbb{X}} \bullet^0 D(L \bullet^0 P_1^{q, q}) \\ &= \eta_{\mathbb{X}} \bullet^0 D(\pi_0(M)) \\ &= M \end{aligned}$$

where the last equality holds by universality of adjunctions. Similarly one gets

$$\eta_{\mathbb{X}} \bullet^0 DL \bullet^0 \mathbb{P}(Q) = N$$

and

$$\eta_{\mathbb{X}} \bullet^0 DL \bullet^0 \mathbb{P}(\phi) = \omega$$

and this concludes the proof. \square

Lemma 6.21. *Sesqui-functor $\Omega^{(n)}$ preserves h -surjective morphisms.*

Proof. This is absolutely straightforward. Let $L : \mathbb{K} \rightarrow \mathbb{A}$ be an h -surjective morphism. Then, for a fixed object κ of \mathbb{K}

$$\mathbb{P}_{\kappa, \kappa}(L) = D(L_1^{\kappa, \kappa})$$

Now $L_1^{\kappa, \kappa}$ is h -surjective by definition since L is, D preserves trivially h -surjective morphisms. \square

Proposition 6.22. *Let the exact sequence of pointed n -groupoids*

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \Downarrow \lambda & \swarrow & \\ \mathbb{A} & \xrightarrow{F} & \mathbb{B} & \xrightarrow{G} & \mathbb{C} \end{array}$$

be given. Then the sequence

$$\begin{array}{ccccc} \Omega \mathbb{A} & \xrightarrow{\Omega F} & \Omega \mathbb{B} & \xrightarrow{\Omega G} & \Omega \mathbb{C} \\ & \searrow & \Downarrow \Omega \lambda & \swarrow & \\ & & 0 & & \end{array}$$

is an exact sequence of pointed n -groupoids.

Proof. Let $L : \mathbb{A} \rightarrow \mathbb{K}$ be the comparison with the kernel of G , that is h -surjective by definition. By *Lemma 6.20* above, ΩL is the comparison with the kernel of ΩG , and it is h -surjective by *Lemma 6.21*. \square

Besides we get the following *for free*

Corollary 6.23. *Sesqui-functor $\pi_1^{(n)}$ preserves exact sequences, reversing the direction of the 2-morphism.*

Similar results hold for the non pointed case, when we fix suitable points.

6.5 Fibration sequence of a n -functor and the Ziqqurath of exact sequences

Purpose of this and next sections is to establish the setting in order to get the main result.

Let be given a morphism of n -groupoids

$$F : \mathbb{B} \rightarrow \mathbb{C}$$

If we fix an object β of \mathbb{B} , the fiber diagram over $F\beta$

$$\begin{array}{ccccc} & & [F\beta] & & \\ & \searrow & \Downarrow \varphi & \swarrow & \\ \mathbb{K} & \xrightarrow{K} & \mathbb{B} & \xrightarrow{F} & \mathbb{C} \end{array} \quad (6.12)$$

is an exact sequence of n -groupoids. In the following sections we will show that this produces a canonical exact sequence

$$\begin{array}{ccccccc} & & [(*, 1_\beta, \beta)] & & [F\beta] & & \\ & \searrow & \Downarrow \sigma & \swarrow & \Downarrow \varphi & \searrow & \\ \mathbb{P}_{\beta, \beta}(\mathbb{B}) & \xrightarrow[\mathbb{P}_{\beta, \beta}(F)]{} & \mathbb{P}_{F\beta, F\beta}(\mathbb{C}) & \xrightarrow{\nabla} & \mathbb{K} & \xrightarrow{K} & \mathbb{B} \xrightarrow{F} \mathbb{C} \\ & & & & \parallel & & \\ & & & & [\beta] & & \end{array}$$

i.e. the sequence represented above is exact in $\mathbb{P}_{F\beta, F\beta}(\mathbb{C})$, in \mathbb{K} and in \mathbb{B} .

6.5.1 Connecting morphism ∇

Although ∇ is easily obtained by means of the universal property of h -pullback of \mathbb{K} , we will consider a slightly more general situation in order to apply induction properly in the construction of the exact sequence.

Let us consider the (past) h -fiber $\mathbb{K} = \mathbb{F}_{F, F\beta}^{(p)}$, $\beta \in \mathbb{B}_0$. Then for any other $\beta' \in \mathbb{B}_0$, the universal property yields a

$$\nabla = \nabla_{F\beta, \beta', F}^{(p)} : \mathbb{P}_{F\beta, F\beta'}(\mathbb{C}) \rightarrow \mathbb{F}_{F, F\beta}^{(p)} = \mathbb{K}$$

$$\begin{array}{ccccc} \mathbb{P}_{F\beta, F\beta'}(\mathbb{C}) & \xrightarrow{\quad} & \mathbb{I}_{(n)} & & \\ \downarrow \nabla & \searrow \varepsilon_C^{F\beta, F\beta'} & \downarrow [\beta'] & & \\ & \mathbb{K} & \xrightarrow{K} & \mathbb{B} & \\ & \downarrow & \nearrow \varphi & \downarrow F & \\ \mathbb{I}_{(n)} & \xrightarrow{id} & \mathbb{I}_{(n)} & \xrightarrow{[F\beta]} & \mathbb{C} \end{array}$$

By Lemma 2.14 this is better understood as

$$\begin{array}{ccc}
 \mathbb{P}_{F\beta, F\beta'}(\mathbb{C}) & \longrightarrow & \mathbb{I}_{(\beta)} \\
 \nabla \downarrow & (\dagger) & \downarrow [\beta'] \\
 \mathbb{K} & \xrightarrow{K} & \mathbb{B} \\
 \downarrow & \nearrow \varphi & \downarrow F \\
 \mathbb{I}_{(\beta)} & \xrightarrow{[F\beta]} & \mathbb{C}
 \end{array} \tag{6.13}$$

Where upper commuting square (\dagger) is a pullback, i.e. ∇ is the strict fiber of $K = K_{F, F\beta}^{(p)}$ over β' .

Proposition 6.24. *The sequence $\langle \nabla, id_{[\beta]}, K \rangle$ above is exact.*

Proof. What we must prove is that the comparison

$$L : \mathbb{P}_{F\beta, F\beta'}(\mathbb{C}) \rightarrow \mathbb{F}_{K, \beta'}^{(f)} = \mathbb{H}$$

is h -surjective. We will prove the statement by induction.

For $n = 1$ the functor

$$D(\mathbb{C}_1(F\beta, F\beta')) \rightarrow \mathbb{F}_{K, \beta'}^{(f)}$$

is clearly essentially surjective on objects and full. In fact it is defined on objects as in the general case L_0 below, hence it is essentially surjective on objects for the same proof. Moreover it is also full: chose two objects c_1, c'_1 in the domain, then we get objects

$$((*, c_1, \beta'), 1_{\beta'}, *), \quad ((*, c'_1, \beta'), 1_{\beta'}, *)$$

An arrow between them is a 1-cell $b_1 : \beta' \rightarrow \beta'$ such that

$$(1) \ c'_1 \circ Fb_1 = c_1 \quad (2) \ b_1 \circ 1_{\beta'} = 1_{\beta'}$$

Condition (2) forces $b_1 = 1_{\beta'}$. This makes condition (1) true only for $c_1 = c'_1$, i.e. the image hom-set is a singleton, implying that functor L on homs is trivially surjective.

Henceforth let us suppose $n > 1$. In order to fix notation we recall the h -fiber \mathbb{H} is a triple (\mathbb{H}, H, ψ) .

Equations

$$L \bullet^0 H = \nabla, \quad L \bullet^0 \varphi = id_{[\beta]}$$

give L on objects:

$$L_0 : (*, F\beta \xrightarrow{c_1} F\beta', *) \mapsto ((*, F\beta \xrightarrow{c_1} F\beta', \beta'), \beta' \xrightarrow{1_{\beta'}} \beta', *)$$

This is indeed an essentially surjective map. In fact, given

$$h_0 = ((*, F\beta \xrightarrow{c_1} Fb_0, b_0), b_0 \xrightarrow{b_1} \beta', *) \in \mathbb{H}_0$$

there exist a $\overline{p_0}$ and a $\overline{h_1} : h_0 \rightarrow L(\overline{p_0})$: simply let

$$\overline{p_0} = (*, F\beta \xrightarrow{c_1} Fb_0 \xrightarrow{Fb_1} F\beta', *)$$

and $\overline{h_1}$ as below

$$\begin{array}{c} h_0 = ((*, F\beta \xrightarrow{c_1} Fb_0, b_0), b_0 \xrightarrow{b_1} \beta', *) \\ \overline{h_1} \downarrow \\ L(\overline{p_0}) = ((*, F\beta \xrightarrow{c_1 Fb_1} Fb_0, b_0), b_0 \xrightarrow{b_1} \beta', *) \end{array} \quad \begin{array}{c} \vdots \\ = \\ \vdots \end{array}$$

Then we fix a pair of objects p_0, p'_0 of $\mathbb{P}_{F\beta, F\beta'}(\mathbb{C})$

$$p_0 = (*, F\beta \xrightarrow{c_1} F\beta', *), \quad p'_0 = (*, F\beta \xrightarrow{c'_1} F\beta', *).$$

As we have shown in proving the universal property of h -pullbacks, the comparison on homs

$$L_1^{p_0, p'_0} : [\mathbb{P}_{F\beta, F\beta'}(\mathbb{C})]_1(p_0, p'_0) \longrightarrow [\mathbb{F}_{K, \beta'}^{(f)}]_1(Fp_0, Fp'_0)$$

is given by universal property on homs (*i.e.* for $(n-1)$ groupoids)

$$L' : \mathbb{P}_{c_1, c'_1}(\mathbb{C}_1(F\beta, F\beta')) \longrightarrow \mathbb{F}_{K_1^{Hh_0, Hh'_0}, c'_1}^{(p)}$$

Now $c_1 = c_1 \circ F_1^{\beta', \beta'}(1_{\beta'})$, and by definition $K_1^{Hh_0, Hh'_0} = K_{c_1 \circ F_1^{\beta', \beta'}, c'_1}^{(f)}$.

Hence we started with a comparison of the kind

$$\mathbb{P}_{x, Fy}(\mathbb{Z}) \longrightarrow \mathbb{F}_{K_{F, x}^{(p)}}^{(f)} \quad (6.14)$$

and we obtain its homs part as a comparison of the kind

$$\mathbb{P}_{Fy, x}(\mathbb{Z}) \longrightarrow \mathbb{F}_{K_{F, x}^{(f)}}^{(p)} \quad (6.15)$$

This situation requires to check also that comparison (6.15) is essentially surjective and then calculate it on homs. We obtain a comparison (6.14)

that terminates a two-level induction process and gives at once that both comparisons are h -surjective.

In fact the same calculation as above shows that (6.15) is essentially surjective, but reverses the direction. Nevertheless this is not a serious obstruction, as all cells of an n -groupoid are equivalences.

In order to understand this fully, it may be interesting to make the construction explicit for 1-cells. Let 1-cell $h_1 : Lp_0 \rightarrow Lp'_0$ as in the following diagram

$$\begin{array}{c} Lp_0 = ((*, F\beta \xrightarrow{c_1} F\beta' \quad \beta') , \beta' \xrightarrow{1_{\beta'}} \beta' , *) \\ \downarrow h_1 \\ Lp'_0 = ((*, F\beta \xrightarrow{c'_1} F\beta', \beta') , \beta' \xrightarrow{1_{\beta'}} \beta' , *) \end{array}$$

$\begin{array}{ccccc} & & \swarrow c_2 & \downarrow Fb_1 & \downarrow b_1 \\ & & & & \swarrow b_2 \end{array}$

i.e.

$$h_1 = ((1_*, c_1 \circ Fb_1 \xRightarrow{c_2} 1_{F\beta} \circ c'_1, b_1), 1_{\beta'} \circ 1_{\beta'} \xRightarrow{b_2} b_1 \circ 1_{\beta'}, 1_*)$$

then there exist a $\overline{p_1} : p_0 \rightarrow p'_0$ and a $\overline{h_2} : L(\overline{p_1}) \rightarrow h_1$. In fact such a $\overline{p_1}$ should be of the form $(*, c_1 \xRightarrow{\overline{c_2}} c'_1, *)$. Hence $L(\overline{p_1})$ is of the form

$$\begin{array}{c} Lp_0 = ((*, F\beta \xrightarrow{c_1} F\beta' \quad \beta') , \beta' \xrightarrow{1_{\beta'}} \beta' , *) \\ \downarrow L(\overline{p_1}) \\ Lp'_0 = ((*, F\beta \xrightarrow{c'_1} F\beta', \beta') , \beta' \xrightarrow{1_{\beta'}} \beta' , *) \end{array}$$

$\begin{array}{ccc} & \swarrow c_2 & \\ & & = \end{array}$

i.e.

$$L(\overline{p_1}) = ((1_*, c_1 \circ 1_{F\beta'} \xRightarrow{\overline{c_2}} 1_{F\beta} \circ c'_1, 1_{\beta'}), id_{1_{\beta'}}, 1_*)$$

Then it suffices to let

$$\overline{c_2} = c_1 \xRightarrow{c_1 Fb_2} c_1 Fb_1 \xRightarrow{c_2} c'_1$$

and get the wanted 2-cell:

$$\begin{array}{c} L(\overline{p_1}) = ((1_*, c_1 \circ 1_{F\beta'} \xRightarrow{(c_1 Fb_2)c_2} 1_{F\beta} \circ c'_1, 1_{\beta'}) , 1_{\beta'} \xRightarrow{id_{1_{\beta'}}} 1_{\beta'} , 1_*) \\ \Downarrow \\ h_1 = ((1_*, c_1 \circ Fb_1 \xRightarrow{c_2} 1_{F\beta} \circ c'_1, b_1) , 1_{\beta'} \xRightarrow{b_2} b_1 , 1_*) \end{array}$$

$\begin{array}{ccccc} & & \downarrow c_1 Fb_2 & \equiv & \downarrow b_2 \\ & & & & \downarrow b_2 \end{array}$

□

6.5.2 Connecting 2-morphism σ

In order to paste diagram (6.13) with \mathbb{P} of the original sequence, exactness in $\mathbb{P}_{F\beta, \mathbb{F}\beta'}(\mathbb{C})$ must be shown, that means we have to find the 2-morphism $[0] \Rightarrow \mathbb{P}(F) \bullet^0 \nabla$ that realizes exactness. This is done by means of 2-dimensional universal property of h -pullbacks.

Let us recognize this fact in the following diagram:

$$\begin{array}{ccccc}
 \mathbb{P}_{\beta, \beta'}(\mathbb{B}) & \xrightarrow{\mathbb{P}(F)} & \mathbb{P}_{F\beta, F\beta'}(\mathbb{C}) & \longrightarrow & \mathbb{I}_{(n)} \\
 \downarrow & \nearrow \sigma & \downarrow \nabla & (pb) & \downarrow [\beta'] \\
 \mathbb{I}_{(n)} & \xrightarrow{[(*, 1_\beta, \beta)]} & \mathbb{K} & \xrightarrow{K} & \mathbb{B} \\
 & & \downarrow & \nearrow \varphi & \downarrow F \\
 & & \mathbb{I}_{(n)} & \xrightarrow{[F\beta]} & \mathbb{C}
 \end{array} \tag{6.16}$$

Construction of σ

We can apply 2-dimensional universal property of $\mathbb{K} = \mathbb{F}_{F, F\beta}^{(p)}$ (even if h -pullbacks regularity is enough) to the following set of data: 2-morphisms

$$\begin{array}{ccc}
 \mathbb{P}(\mathbb{B}) \xrightarrow{[\beta]} \mathbb{B} & \mathbb{P}(\mathbb{B}) \xrightarrow{[\beta']} \mathbb{B} & \mathbb{P}(\mathbb{B}) \xrightarrow{[\beta]} \mathbb{B} \\
 \downarrow \text{\scriptsize{id}} \downarrow F & \downarrow \text{\scriptsize{\varepsilon}_{\mathbb{B}} F} \downarrow F & \downarrow \text{\scriptsize{\varepsilon}_{\mathbb{B}}} \downarrow F \\
 \mathbb{I} \xrightarrow{[F\beta]} \mathbb{C} & \mathbb{I} \xrightarrow{[F\beta]} \mathbb{C} & \mathbb{I} \xrightarrow{[F\beta]} \mathbb{C}
 \end{array}$$

over the base, 2-morphism $\left(\begin{smallmatrix} \mathbb{P}(\mathbb{B}) \\ \text{\scriptsize{=}} \\ \mathbb{I} \end{smallmatrix} \right)$ and $[\beta] \left(\begin{smallmatrix} \xrightarrow{\varepsilon_{\mathbb{B}}} \\ \downarrow \\ \mathbb{I} \end{smallmatrix} \right) [\beta']$

and (identity) 3-morphism $\left\| \begin{array}{c} [F\beta] = [\beta] F \\ \text{\scriptsize{=}} \\ [F\beta] \xrightarrow{\varepsilon_{\mathbb{B}} F} [\beta'] F \end{array} \right\|$, where the second diagram is

justified by the equalities

$$\mathbb{P}(F) \bullet^0 \nabla \bullet^0 \varphi = \mathbb{P}(F) \bullet^0 \varepsilon_{\mathbb{C}} = \varepsilon_{\mathbb{B}} \bullet^0 F.$$

Then there exists a unique $\sigma : [(*, 1_\beta, \beta)] \Rightarrow \mathbb{P}(F) \bullet^0 \nabla$ such that

$$(i) \quad \sigma \bullet^0 K = \varepsilon_{\mathbb{B}}^{\beta, \beta'} \quad (ii) \quad \sigma * \varphi = id_{\varepsilon_{\mathbb{B}}^{\beta, \beta'} \bullet^0 F}$$

Comparison morphism

Proposition 6.25. *The triple $(\mathbb{P}(F), \sigma, \nabla)$ is exact.*

Proof. In order to show that triple $(\mathbb{P}(F), \sigma, \nabla)$ is exact, we must verify that comparison with (past) h -fibre of ∇ is h -surjective. Namely we construct the h -pullback

$$\begin{array}{ccc}
 \mathbb{N} & \xrightarrow{J} & \mathbb{P}_{F\beta, F\beta'}(\mathbb{C}) \\
 \downarrow & \nearrow \xi & \downarrow \nabla \\
 \mathbb{I} & \xrightarrow{[(*, 1_{F\beta}, \beta)]} & \mathbb{K}
 \end{array}
 \quad \text{where} \quad
 \begin{aligned}
 \mathbb{K} &= \mathbb{F}_{F, F\beta}^{(p)} \\
 \nabla &= \nabla_{F\beta, \beta', F}^{(p)} \\
 \mathbb{N} &= \mathbb{F}_{(*, 1_{F\beta}, \beta), \nabla}^{(p)}
 \end{aligned}$$

The universal property for \mathbb{N} yields a unique $N : \mathbb{P}_{\beta, \beta'}(\mathbb{B}) \rightarrow \mathbb{N}$ such that

$$(i) \ N \bullet^0 J = \mathbb{P}(F) \quad (ii) \ N \bullet^0 \xi = \sigma \quad (6.17)$$

For $n = 1$ the result is easily obtained. In fact in this case \mathbb{N} is a discrete groupoid, and since also $\mathbb{P}_{\beta, \beta'}(\mathbb{B})$ is, it suffices to check surjectivity on objects, and this is achieved in the same manner as h -surjectivity for the general case.

Hence let us suppose $n > 1$.

We will need an explicit description of n -functor $N = \langle N_0, N_1^-, \cdot \rangle$.

$$N_0 : (*, \beta \xrightarrow{b_1} \beta', *) \mapsto (*, (1_*, Fb_1 \xRightarrow{id_{Fb_1}} Fb_1, b_1), (*, Fb_1, *))$$

Moreover for any pair of objects $(*, b_1, *)$ and $(*, b'_1, *)$ in $\mathbb{P}_{\beta, \beta'}(\mathbb{B})$,

$$N_1^{(*, b_1, *), (*, b'_1, *)}$$

is obtained by the universal property of h -pullbacks as shown by the diagram below, where $p_0 = (*, b_1, *)$ and $p'_0 = (*, b'_1, *)$,

$$\begin{array}{ccc}
 [\mathbb{P}_{\beta, \beta'}(\mathbb{B})]_1((*, b_1, *), (*, b'_1, *)) & \xrightarrow{[\mathbb{P}(F)]_1^{(*, b_1, *), (*, b'_1, *)}} & [\mathbb{P}_{F\beta, F\beta'}(\mathbb{C})]_1((*, Fb_1, *), (*, Fb'_1, *)) \\
 \downarrow N_1^{(*, b_1, *), (*, b'_1, *)} & & \downarrow [\nabla]_1^{(*, Fb_1, *), (*, Fb'_1, *)} \\
 \mathbb{N}_1(Np_0, Np'_0) & \xrightarrow{J_1^{Np_0, Np'_0}} & \mathbb{K}_1((*, Fb_1, \beta'), (*, Fb'_1, \beta')) \\
 \downarrow & \nearrow \xi_1^{Np_0, Np'_0} & \downarrow (1_*, 1_{Fb_1}, b_1) \circ - \\
 \mathbb{I}_{(n-1)} & \xrightarrow{[(1_*, 1_{Fb'_1}, b'_1)]} & \mathbb{K}_1((*, 1_{F\beta}, \beta), (*, Fb'_1, \beta'))
 \end{array}
 \quad (6.18)$$

such that

$$(i) N_1^{\diamond, \diamond} \bullet^0 J_1^{\diamond, \diamond} = [\mathbb{P}(F)]_1^{\diamond, \diamond} \quad (ii) N_1^{\diamond, \diamond} \bullet^0 \xi_1^{\diamond, \diamond} = \sigma_1^{\diamond, \diamond} \quad (6.19)$$

Claim 6.26. N is essentially surjective on objects.

Let an object n_0 of \mathbb{N} be given

$$n_0 = (*, (1_*, Fb_1 \xrightarrow{c_2} c_1, b_1), (*, F\beta \xrightarrow{c_1} F\beta', *))$$

then it suffices to consider the object $p_0 = (*, b_1, *)$ to get an arrow $N(p_0) \rightarrow n_0$, as suggested by the diagram below

$$\begin{array}{ccc} N(p_0) & = & (*, (*, Fb_1 \xrightarrow{id} Fb_1, b_1), (*, F\beta \xrightarrow{Fb_1} F\beta', *)) \\ \downarrow & & \vdots \quad \vdots \quad \vdots \quad \equiv \quad \vdots \quad \vdots \quad \swarrow c_2 \quad \vdots \quad \vdots \\ n_0 & = & (*, (*, Fb_1 \xrightarrow{c_2} c_1, b_1), (*, F\beta \xrightarrow{c_1} F\beta', *)) \end{array}$$

Claim 6.27. For any pair $(*, b_1, *)$, $(*, b'_1, *)$, the $(n-1)$ -functor $N_1^{(*, b_1, *), (*, b'_1, *)}$ is h -surjective.

Here comes the inductive step: we want to get $N_1^{\diamond, \diamond}$ from a situation in which is indeed a comparison itself, as in the original setting of diagram (6.16). To this end let us consider the diagram

$$\begin{array}{ccccc} \mathbb{P}_{b_1, b'_1}(\mathbb{B}_1(\beta, \beta')) & \xrightarrow{[\mathbb{P}(F)]_1^{\diamond, \diamond}} & \mathbb{P}_{Fb_1, Fb'_1}(\mathbb{C}_1(F\beta, F\beta')) & \xrightarrow{\quad} & \mathbb{I} \\ \downarrow & \swarrow \sigma_1^{\diamond, \diamond} & \downarrow (1_*, id_{Fb_1}, b_1) \circ [\nabla]_1^{\diamond, \diamond} & \quad (\dagger) & \downarrow [b_1] \\ \mathbb{I} & \xrightarrow{[(1_*, id_{Fb'_1}, b'_1)]} & \mathbb{K}_1((*, 1_{F\beta}, \beta), (*, Fb'_1, \beta')) & \xrightarrow{K_1^{\diamond, \diamond}} & \mathbb{B}_1(\beta, \beta') \\ & & \downarrow & \swarrow \varphi_1^{\diamond, \diamond} & \downarrow F_1^{\beta, \beta'} \\ & & \mathbb{I} & \xrightarrow{[Fb'_1]} & \mathbb{C}_1(F\beta, F\beta') \end{array}$$

By definition of h -pullback the square $\boxed{\varphi_1^{\diamond, \diamond}}$ is the h -fiber $\mathbb{F}_{F_1^{\beta, \beta'}, Fb_1}^{(f)}$. Furthermore the square (\dagger) is a pullback by *Lemma 2.14*. In fact this follows by the universal property, as the following equations hold:

$$\begin{aligned} (\clubsuit) \quad & ((1_*, id_{Fb_1}, b_1) \circ [\nabla]_1^{(*, Fb_1, *), (*, Fb'_1, *)}) \bullet^0 K_1^{(*, 1_{F\beta}, \beta), (*, Fb'_1, \beta')} = [b_1] \\ (\diamondsuit) \quad & ((1_*, id_{Fb_1}, b_1) \circ [\nabla]_1^{(*, Fb_1, *), (*, Fb'_1, *)}) \bullet^0 \varphi_1^{(*, 1_{F\beta}, \beta), (*, Fb'_1, \beta')} = \varepsilon_{\mathbb{C}_1(F\beta, F\beta')}^{Fb_1, Fb'_1} \end{aligned}$$

proof of (\clubsuit) and of (\diamond) . They follow easily from the universal property in dimension $n - 1$:

$$\begin{aligned}
 ((1_*, id_{Fb_1}, b_1) \circ \nabla_1^{\diamond, \diamond}) \bullet^0 K_1^{\diamond, \diamond} &= K((1_*, id_{Fb_1}, b_1)) \circ (\nabla_1^{\diamond, \diamond} \bullet^0 K_1^{\diamond, \diamond}) \\
 &= K((1_*, id_{Fb_1}, b_1)) \circ ([\nabla \bullet^0 K]_1^{\diamond, \diamond}) \\
 &\stackrel{(*)}{=} K((1_*, id_{Fb_1}, b_1)) \circ ([\beta']_1^{\diamond, \diamond}) \\
 &= [K((1_*, id_{Fb_1}, b_1))] \\
 &= [b_1]
 \end{aligned}$$

$$\begin{aligned}
 ((1_*, id_{Fb_1}, b_1) \circ [\nabla]_1^{\diamond, \diamond}) \bullet^0 \varphi_1^{\diamond, \diamond} &\stackrel{(i)}{=} \left(1_{Fb_1} \circ (\nabla_1^{\diamond, \diamond} \bullet^0 KF_1^{\diamond, \diamond})\right) \bullet^1 \left(1_{F\beta} \circ (\nabla_1^{\diamond, \diamond} \bullet^0 \varphi_1^{\diamond, \diamond})\right) \\
 &= \left(\nabla_1^{\diamond, \diamond} \bullet^0 KF_1^{\diamond, \diamond}\right) \bullet^1 \left(\nabla_1^{\diamond, \diamond} \bullet^0 \varphi_1^{\diamond, \diamond}\right) \\
 &\stackrel{(ii)}{=} \nabla_1^{\diamond, \diamond} \bullet^0 \varphi_1^{\diamond, \diamond} \\
 &\stackrel{(*)}{=} [\varepsilon_{\mathbb{C}}^{F\beta, F\beta'}]_1^{Fb_1, Fb'_1} \\
 &= \varepsilon_{\mathbb{C}_1(F\beta, F\beta')}^{Fb_1, Fb'_1}
 \end{aligned}$$

where equations marked $(*)$ hold for inductive definition of universal property of h -pullbacks, while (i) is composition axiom of 2-morphism, and (ii) since the first component of 1-composition is a 1-morphism.

Hence

$$(1_*, id_{Fb_1}, b_1) \circ [\nabla_{F\beta, \beta', F}^{(p)}]_1^{\diamond, \diamond} = \nabla_{F_1^{\beta, \beta'}, b'_1, b_1, F_1^{\beta, \beta'}}^{(f)}$$

i.e. it is a “ ∇ ” itself.

In other words, we wanted to prove that comparison with an h -fiber of

$$\nabla_{F\beta, \beta', F}^{(p)} : \mathbb{P}_{F\beta, F\beta'}(\mathbb{C}) \longrightarrow \mathbb{F}_{F, F\beta}^{(p)}$$

is h -surjective, and we find out that is equivalent to asking:

1. its essential surjectivity
2. that the *comparison* in dimension $(n-1)$ w.r.t.

$$\nabla_{F_1^{\beta, \beta'}, b'_1, b_1, F_1^{\beta, \beta'}}^{(f)} : \mathbb{P}_{Fb_1, Fb'_1}(\mathbb{C}_1(F\beta, F\beta')) \longrightarrow \mathbb{F}_{F_1^{\beta, \beta'}, F_1^{\beta, \beta'}, b'_1}^{(f)}$$

is h -surjective.

In conclusion, we get the same construction as in dimension n , up to directions.

Since we are in a (weakly) invertible setting, we can apply induction properly and get the result. \square

6.5.3 The fibration sequence of F

From now on we will consider h -kernels in the pointed setting $n\mathbf{Gpd}_*$. Nevertheless all the constructions plainly apply to h -fibers in $n\mathbf{Gpd}$, as shown in the construction for diagram (6.12).

Let $F : \mathbb{C} \rightarrow \mathbb{D}$ be a morphism of pointed n -groupoids. Specializing constructions above with $\beta = *$ and $0 = [*]$ we can exhibit the exact sequence

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & \searrow & \Downarrow \sigma & \searrow & \Downarrow \varphi & \searrow & \\
 \Omega \mathbb{B} & \xrightarrow{\Omega F} & \Omega \mathbb{C} & \xrightarrow{\nabla} & \mathbb{K} & \xrightarrow{K} & \mathbb{B} \xrightarrow{F} \mathbb{C} \\
 & \nearrow & & \nearrow & & \nearrow & \\
 & & 0 & & 0 & &
 \end{array}$$

Since Ω preserves exactness, this gives another exact sequence

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & \searrow & \Downarrow \Omega \sigma & \searrow & \Downarrow \Omega \varphi & \searrow & \\
 \Omega^2 \mathbb{B} & \xrightarrow{\Omega^2 F} & \Omega^2 \mathbb{C} & \xrightarrow{\Omega \nabla} & \Omega \mathbb{K} & \xrightarrow{\Omega K} & \Omega \mathbb{B} \xrightarrow{\Omega F} \Omega \mathbb{C} \\
 & \nearrow & & \nearrow & & \nearrow & \\
 & & 0 & & 0 & &
 \end{array}$$

Those can be pasted together in the seven-term exact sequence

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & \searrow & \Downarrow \Omega \sigma & \searrow & \Downarrow \Omega \varphi & \searrow & \Downarrow \sigma & \searrow & \Downarrow \varphi & \searrow & \\
 \Omega^2 \mathbb{B} & \xrightarrow{\Omega^2 F} & \Omega^2 \mathbb{C} & \xrightarrow{\Omega \nabla} & \Omega \mathbb{K} & \xrightarrow{\Omega K} & \Omega \mathbb{B} & \xrightarrow{\Omega F} & \Omega \mathbb{C} & \xrightarrow{\nabla} & \mathbb{K} \xrightarrow{K} \mathbb{B} \xrightarrow{F} \mathbb{C} \\
 & \nearrow & & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\
 & & 0 & & 0 & & 0 & & 0 & &
 \end{array}$$

Of course the process can be iterated indefinitely in order to get a longer exact sequence, even if it trivializes after n applications.

6.5.4 The Ziqqurath of a morphism of (pointed) n -groupoids

A different perspective is gained by considering sesqui-functor π_1 in place of Ω .

In fact in the longer exact sequences obtained above, repeated applications of Ω give structures which are discrete in higher dimensional cells. Their exactness may be fruitfully investigated in lower dimensional settings, *i.e.* after repeated applications of π_0 .

To this end we state the following

Lemma 6.28. *Sesqui-functor π_0 commutes with sesqui-functor π_1 , i.e. for every integer $n > 1$ the following diagram is commutative*

$$\begin{array}{ccc} n\mathbf{Gpd}_* & \xrightarrow{\pi_0^{(n)}} & (n-1)\mathbf{Gpd}_* \\ \pi_1^{(n)} \downarrow & & \downarrow \pi_1^{(n-1)} \\ (n-1)\mathbf{Gpd}_* & \xrightarrow{\pi_0^{(n-1)}} & (n-2)\mathbf{Gpd}_* \end{array}$$

Proof. This can be proved directly. For $n = 2$ the diagram commutes trivially. Hence let us suppose $n > 2$. Let us be given a pointed n -groupoid \mathbb{C} , then by direct application of inductive definitions involved one has

$$\begin{aligned} [\pi_0(\pi_1(\mathbb{C}))]_0 &= [\pi_0(\mathbb{C}_1(*, *))_0]_0 \\ &= [\mathbb{C}_1(*, *)]_0 \\ &= [\pi_0(\mathbb{C}_1(*, *))_0]_0 \\ &= [[\pi_0(\mathbb{C})]_1(*, *)]_0 \\ &= [\pi_1(\pi_0(\mathbb{C}))]_0 \end{aligned}$$

Moreover for any pair of “objects” c_1, c'_1 one has

$$\begin{aligned} [\pi_0(\pi_1(\mathbb{C}))]_1(c_1, c'_1) &= [\pi_0(\mathbb{C}_1(*, *))_1(c_1, c'_1)]_0 \\ &= \pi_0([\mathbb{C}_1(*, *)]_1(c_1, c'_1)) \\ &= [\pi_0(\mathbb{C}_1(*, *))_1(c_1, c'_1)]_0 \\ &= [[\pi_0(\mathbb{C})]_1(*, *)]_1(c_1, c'_1) \\ &= [\pi_1(\pi_0(\mathbb{C}))]_1(c_1, c'_1) \end{aligned}$$

Finally this extends plainly to morphisms and 2-morphisms. □

Remark 6.29. In the language of loops, we can re-state *Lemma* above in other terms:

$$\pi_0(\pi_0(\Omega(-))) = \pi_0(\Omega(\pi_0(-)))$$

Let now a morphism $F : \mathbb{C} \rightarrow \mathbb{D}$ of pointed n -groupoids be given. Then the h -kernel exact sequence

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \Downarrow \varphi & \swarrow & \\ \mathbb{K} & \xrightarrow{K} & \mathbb{B} & \xrightarrow{F} & \mathbb{C} \end{array}$$

gives two exact sequences of pointed $(n-1)$ -groupoids:

$$\begin{array}{ccc} \pi_1 \mathbb{K} & \xrightarrow{\pi_1 K} & \pi_1 \mathbb{B} \xrightarrow{\pi_1 F} \pi_1 \mathbb{C} \\ & \searrow \downarrow \pi_1 \varphi & \nearrow \\ & 0 & \end{array} \quad \begin{array}{ccc} & & 0 \\ & \searrow \downarrow \pi_0 \varphi & \swarrow \\ \pi_0 \mathbb{K} & \xrightarrow{\pi_0 K} & \pi_0 \mathbb{B} \xrightarrow{\pi_0 F} \pi_0 \mathbb{C} \end{array}$$

Those can be connected together in order to give a six term exact sequence of pointed $(n - 1)$ -groupoids

$$\begin{array}{ccccccc}
 & & & 0 & & & 0 \\
 & & & \Downarrow \delta & & & \Downarrow \pi_0 \varphi \\
 \pi_1 \mathbb{K} & \xrightarrow{\pi_1 K} & \pi_1 \mathbb{B} & \xrightarrow{\pi_1 F} & \pi_1 \mathbb{C} & \xrightarrow{\Delta} & \pi_0 \mathbb{K} & \xrightarrow{\pi_0 K} & \pi_0 \mathbb{B} & \xrightarrow{\pi_0 F} & \pi_0 \mathbb{C} \\
 & & & \Downarrow \pi_1 \varphi & & & \parallel & & & & \\
 & & & 0 & & & 0 & & & &
 \end{array}$$

where $\Delta = \pi_0(\nabla)$ and $\delta = \pi_0(\sigma)$. Notice that the three leftmost terms are endowed with strict monoidal structure and weak inverses.

Applying π_0 and π_1 , we get two six-term exact sequences. Those can be pasted by *Lemma 6.28* in a nine-term exact sequence of $(n - 2)$ -groupoids (cells to be pasted are dotted in the diagram):

$$\begin{array}{c}
 \begin{array}{ccccccc}
 \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot \\
 & \searrow & \downarrow \pi_1^2 \varphi & \nearrow & \downarrow \pi_1 \delta & \searrow & \downarrow \pi_1 \pi_0 \varphi \\
 & & \cdot & & \cdot & & \cdot
 \end{array} \\
 \begin{array}{ccccccc}
 \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot \\
 & \searrow & \downarrow \pi_0 \delta & \nearrow & \downarrow \pi_0 \pi_1 \varphi & \searrow & \downarrow \pi_0^2 \varphi \\
 & & \cdot & & \cdot & & \cdot
 \end{array}
 \end{array}$$

Now the three leftmost terms are endowed with a commutative strict monoidal structure and weak inverses and the three middle terms are endowed with strict monoidal structure and weak inverses.

Iterating the process we obtain a sort of tower, a Ziqqurath, in which the lower is the level, the lower is the dimension the longer is the length of the sequence.

$$\begin{array}{ccc}
 \begin{array}{c} \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \end{array} & & n\mathbf{Gpd} \\
 \begin{array}{c} \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \end{array} & & (n - 1)\mathbf{Gpd} \\
 \begin{array}{c} \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \end{array} & & (n - 2)\mathbf{Gpd} \\
 \vdots & & \\
 \begin{array}{c} \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \end{array} \quad \dots \quad \begin{array}{c} \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \end{array} & & \mathbf{Gpd} \\
 \begin{array}{c} \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \end{array} \quad \dots \quad \begin{array}{c} \cdot \xrightarrow{\quad} \cdot \xrightarrow{\quad} \cdot \end{array} & & \mathbf{Set}
 \end{array}$$

In particular, the last row counts $3 \cdot n$ terms. From left to right, there are $3 \cdot n - 6$ abelian groups, 3 groups and 3 pointed sets.

The row before the last counts $3 \cdot (n - 1)$ terms. From left to right, there are $3 \cdot n - 9$ strictly commutative categorical groups, 3 categorical groups, 3 pointed groupoids. Let us observe that categorical groups produced in this way are strict monoidal weakly invertible ones.

Appendix A

n -Groupoids, comparing definitions

A.1 $n\text{Cat}$: the globular approach

In this section we compare the classical globular definition of n -category with the inductively enriched one presented up to here.

The following definition is freely adapted from the one presented in [BH81]. It is essentially the same presented also in [KV91], and is indeed equivalent to that of [Str87].

Definition A.1. *A n -category is a reflexive n -truncated globular set*

$$(\mathcal{C}_\bullet = \{\mathcal{C}_i\}_{i=0,\dots,n}, \{s_i, t_i : \mathcal{C}_{i+1} \rightarrow \mathcal{C}_i\}_i, \{e_{i+1} : \mathcal{C}_i \rightarrow \mathcal{C}_{i+1}\}_i)$$

We will often (ab)use the notations

$$\begin{aligned} s_k : \mathcal{C}_i &\rightarrow \mathcal{C}_k && \text{meaning } s_{i-1} \cdot s_{i-2} \cdot \dots \cdot s_k, \\ t_k : \mathcal{C}_i &\rightarrow \mathcal{C}_k && \text{meaning } t_{i-1} \cdot t_{i-2} \cdot \dots \cdot t_k, \\ e_i : \mathcal{C}_k &\rightarrow \mathcal{C}_i && \text{meaning } e_{k+1} \cdot e_{k+2} \cdot \dots \cdot e_i. \end{aligned}$$

\mathcal{C}_\bullet is endowed with operations $(m < i)$

$$\star^m : \mathcal{C}_i \times_{t_m} \mathcal{C}_i \rightarrow \mathcal{C}_i$$

such that:

1. for all $c, c' \in \mathcal{C}_\bullet$ and $m \leq k$,

$$s_k(c \star^m c') = \begin{cases} s_k(c) & \text{if } m = k \\ s_k(c) \star^m s_k(c') & \text{if } m < k \end{cases}$$

$$t_k(c \star^m c') = \begin{cases} t_k(c') & \text{if } m = k \\ t_k(c) \star^m t_k(c') & \text{if } m < k \end{cases}$$

2. for all $c \in \mathcal{C}_i$, all m ,

$$e_i(s_m(c)) \star^m c = c = c \star^m e_i(t_m(c))$$

3. for all $c, c' \in \mathcal{C}$, all m ,

$$e(c \star^m c') = e(c) \star^m e(c')$$

4. for all $c, c', c'' \in \mathcal{C}$, all m ,

$$c \star^m (c' \star^m c'') = (c \star^m c') \star^m c''$$

5. for all $c, c', d, d' \in \mathcal{C}$, all $p < q$,

$$(c \star^q c') \star^p (d \star^q d') = (c \star^p d) \star^q (c' \star^p d')$$

In order to avoid confusion, this will be called globular n -category.

Given a n -category \mathbb{C} , this defines a globular n -category \mathcal{C} .

In fact it suffices to let

$$\mathcal{C}_0 = \mathbb{C}_0$$

and for every $0 < i \leq n$,

$$\mathcal{C}_i = \bigcup_{c_{i-1}, c'_{i-1} \in \mathcal{C}_{i-1}} [\mathbb{C}_i(c_{i-1}, c'_{i-1})]_0$$

Sources targets and identities are obtained composing the following ones:

$$s_i : \mathcal{C}_{i+1} \rightarrow \mathcal{C}_i, \quad t_i : \mathcal{C}_{i+1} \rightarrow \mathcal{C}_i, \quad e_{i+1} : \mathcal{C}_i \rightarrow \mathcal{C}_{i+1}$$

where

$$s_{i-1}(c_i : c_{i-1} \multimap c'_{i-1}) = c_{i-1}$$

$$t_{i-1}(c_i : c_{i-1} \multimap c'_{i-1}) = c'_{i-1}$$

$$e_{i+1}(c_i) = \left[\mathbb{C}_i(c_{i-1}, c'_{i-1}) u(c_i) \right] (*) : c_i \multimap_{i+1} c_i$$

A simple calculation shows that this forms a reflexive n -truncated globular set.

Let now suppose we are given a pair $(c_i, c'_i) \in \mathcal{C}_i \times_{t_m \times s_m} \mathcal{C}_i$.

This means that c_i is a cell of $\mathbb{C}_{m+1}(c_m, c'_m)$ and that c'_i is a cell of $\mathbb{C}_{m+1}(c'_m, c''_m)$, for certain c_m, c'_m, c''_m . Then we can easily define the composition

$$c_i \star^m c'_i = c_i \circ^m c'_i$$

where \circ^m is the m -composition morphism

$$(-) \circ_{c_m, c'_m, c''_m}^m (-) : \mathbb{C}_{m+1}(c_m, c'_m) \times \mathbb{C}_{m+1}(c'_m, c''_m) \rightarrow \mathbb{C}_{m+1}(c_m, c''_m) \quad (\text{A.1})$$

This can be seen as 0-composition. In fact, by the inductive definition of a n -category, there exist c_{m-1}, c'_{m-1} such that $c_m, c'_m, c''_m : c_{m-1} \xrightarrow{m} c'_{m-1}$. Then \circ^m is indeed

$$(-) \circ_{c_m, c'_m, c''_m}^m (c_{m-1}, c'_{m-1}) \circ_{c_m, c'_m, c''_m}^0 (-) : \mathbb{C}_{m+1}(c_m, c'_m) \times \mathbb{C}_{m+1}(c'_m, c''_m) \rightarrow \mathbb{C}_{m+1}(c_m, c''_m)$$

where, as usual, the various $\mathbb{C}_{m+1}(x, y)$ are indeed the short form for $[\mathbb{C}_m(c_{m-1}, c'_{m-1})]_1(x, y)$.

These data satisfy axioms for a globular n -category.

Proof. The proof is divided into five parts, according to the five axioms.

1. The statement will be proved by (finite) induction over k , for a fixed m .

The base of the induction is given by definition:

$$s_{n-1}(c_n \star^{n-1} c'_n) = s_{n-1}(c_{n-1} \xrightarrow{c_n} c'_{n-1} \xrightarrow{c'_n} c''_{n-1}) = c_{n-1}.$$

Now suppose $k \geq m$. Then for every $i > k$ one has

$$\begin{aligned} s_k(c_i \star^m c'_i) &\stackrel{(i)}{=} s_k(s_{k+1}(c_i \star^m c'_i)) \\ &\stackrel{(ii)}{=} s_k(s_{k+1}(c_i) \star^m s_{k+1}(c'_i)) \\ &\stackrel{(iii)}{=} s_k(c_{k+1} \star^m c'_{k+1}) \end{aligned}$$

where (i) holds by definition, (ii) by induction, (iii) is just a typographical substitution.

Indeed what we mean with the expression “ $c_{k+1} \star^m c'_{k+1}$ ” is the image under the morphism (A.1) of the pair (c_{k+1}, c'_{k+1}) . This is inductively defined on homs, hence for strictly $k > m$ one can make it explicit:

$$(c_{k+1}, c'_{k+1}) : (c_k, \bar{c}_k) \xrightarrow{k+1} (c'_k, \bar{c}'_k)$$

This is a $(k-m)$ -cell of $\mathbb{C}_m(c_m, c'_m) \times \mathbb{C}_m(c'_m, c''_m)$, and its image under $[\circ_{c_m, c'_m, c''_m}^m]$ is indeed its image under

$$[\circ_{c_m, c'_m, c''_m}^m]_{k-m}^{(c_k, \bar{c}_k), (c'_k, \bar{c}'_k)}.$$

Then functoriality over the two components of a product implies

$$s_k(c_{k+1} \star^m c'_{k+1}) = s_k(c_{k+1}) \star^m s_k(c'_{k+1}) = s_k(c_i) \star^m s_k(c'_i).$$

Differently for $k = m$,

$$s_k(c_{k+1} \star^k c'_{k+1}) = s_k(c_k \xrightarrow{c_{k+1}} c'_k \xrightarrow{c'_{k+1}} c''_k) = c_k.$$

The analogous statement relative to targets is dealt similarly.

2. First we observe that for every $c_k : c_{k-1} \multimap c'_{k-1}$, $k < n$, functoriality w.r.t. units forces the morphism $\mathbb{C}_k(c_{k-1}, c'_{k-1})u$ to satisfy the equation expressed by the following diagram

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{id} & \mathbb{I} \\ & \searrow [\mathbb{C}_{k+1}(c_k, c_k)u(1_{c_k})] & \downarrow \mathbb{C}_k(c_{k-1}, c'_{k-1})u(c_k)]_1^{*,*} \\ & & \mathbb{C}_{k+1}(c_k, c_k) \end{array}$$

that is

$$[\mathbb{C}_k(c_{k-1}, c'_{k-1})u(c_k)]_1^{*,*} = \mathbb{C}_{k+1}(c_k, c_k)u(1_{c_k})$$

This fact inductively extends and gives relations between units. In particular for $k < i$ this implies

$$\begin{aligned} e_i(c_k) &= e_i(e_{i-1}(\cdots e_{k+1}(e_k(c_k)) \cdots)) \\ &= e_i(e_{i-1}(\cdots e_{k+1}([\mathbb{C}_k(c_{k-1}, c'_{k-1})u(c_k)] \cdots)) \\ &= e_i(e_{i-1}(\cdots e_{k+1}(1_{c_k}) \cdots)) \\ &= e_i(e_{i-1}(\cdots [\mathbb{C}_{k+1}(c_k, c_k)u(1_{c_k})] \cdots)) \\ &= e_i(e_{i-1}(\cdots [\mathbb{C}_k(c_{k-1}, c'_{k-1})u(c_k)]_1 \cdots)) \\ &\quad \vdots \\ &= [\mathbb{C}_k(c_{k-1}, c'_{k-1})u(c_k)]_{i-k} \end{aligned}$$

Hence we can calculate

$$e_i(s_k(c_i)) \star^m c_i = [\mathbb{C}_k(c_{k-1}, c'_{k-1})u(s_k(c_i))]_{i-k} \circ^m c_i = c_i$$

as a direct consequence of neutral m -units.

The analogous statement relative to targets is dealt similarly.

3. For $m < i < n$ we want to prove

$$e_{i+1}(c_i \star^m \bar{c}_i) = e_{i+1}(c_i) \star^m e_{i+1}(\bar{c}_i)$$

or more simply

$$1_{(c_i \circ^m \bar{c}_i)} = 1_{c_i} \circ^m 1_{\bar{c}_i}.$$

This is just functoriality w.r.t. units of the morphism $(-) \circ^m (-)$. In fact if $c_i : c_{i-1} \rightarrow c'_{i-1}$ and $\bar{c}_i : \bar{c}_{i-1} \rightarrow \bar{c}'_{i-1}$, m -composition sends the identity

$$(1_{c_i}, 1_{\bar{c}_i}) : (c_i, \bar{c}_i) \longrightarrow (c_i, \bar{c}_i)$$

of $\mathbb{C}_i(c_{i-1}, c'_{i-1}) \times \mathbb{C}_i(\bar{c}_{i-1}, \bar{c}'_{i-1})$ to the identity

$$1_{(c_i \circ^m \bar{c}_i)} : (c_i \circ^m \bar{c}_i) \longrightarrow (c_i \circ^m \bar{c}_i)$$

of $\mathbb{C}_i(c_{i-1} \circ^m \bar{c}_{i-1}, c'_{i-1} \circ^m \bar{c}'_{i-1})$.

4. Let $m < i \leq n$. We have to prove the equality

$$c_i \star^m (c'_i \star^m c''_i) = (c_i \star^m c'_i) \star^m c''_i$$

i.e.

$$c_i \circ^m (c'_i \circ^m c''_i) = (c_i \circ^m c'_i) \circ^m c''_i$$

This holds by associativity of m -composition.

5. Let us suppose $p < q < i \leq n$. We have to prove the equality

$$(c_i \star^q \bar{c}_i) \star^p (d_i \star^q \bar{d}_i) = (c_i \star^p d_i) \star^q (\bar{c}_i \star^p \bar{d}_i)$$

i.e.

$$(c_i \circ^q \bar{c}_i) \circ^p (d_i \circ^q \bar{d}_i) = (c_i \circ^p d_i) \circ^q (\bar{c}_i \circ^p \bar{d}_i)$$

To this end let us fix notation:

$$\begin{aligned} c_i & : \dots c_q \longrightarrow c'_q : \dots c_p \longrightarrow c'_p \\ \bar{c}_i & : \dots c'_q \longrightarrow c''_q : \dots c_p \longrightarrow c'_p \\ d_i & : \dots d_q \longrightarrow d'_q : \dots c'_p \longrightarrow c''_p \\ \bar{d}_i & : \dots d'_q \longrightarrow d''_q : \dots c'_p \longrightarrow c''_p \end{aligned}$$

The p -composition

$$\circ^p : \mathbb{C}_{p+1}(c_p, c'_p) \times \mathbb{C}_{p+1}(c'_p, c''_p) \rightarrow \mathbb{C}_{p+1}(c_p, c''_p)$$

is functorial w.r.t. all q -compositions, with $p < q$. Indeed q -compositions in the product $\mathbb{C}_{p+1}(c_p, c'_p) \times \mathbb{C}_{p+1}(c'_p, c''_p)$ are products of q -compositions in the components. Hence the pair $(c_i \circ^q \bar{c}_i, d_i \circ^q \bar{d}_i)$ is really a composition $(c_i, d_i) \circ^q (\bar{c}_i, \bar{d}_i)$ and its image under $(-) \circ^p (-)$, namely $(c_i \circ^q \bar{c}_i) \circ^p (d_i \circ^q \bar{d}_i)$, must be equal to the q -composition of the images of the p -composites $c_i \circ^p d_i$ and $\bar{c}_i \circ^p \bar{d}_i$, namely $(c_i \circ^p d_i) \circ^q (\bar{c}_i \circ^p \bar{d}_i)$.

□

Vice-versa a globular n -category \mathcal{C} univocally defines a n -category \mathbb{C} .

(*Idea of a proof.* The degenerate case $n = 0$ gives immediately a (0-truncated globular) set.

For $n = 1$ *Definition A.1* is precisely the definition of a category as a 1-truncated globular set.

So let us suppose $n > 1$. We define a n -category \mathbb{C} in the following way.

\mathbb{C}_0 is the set \mathcal{C}_0 of \mathcal{C} .

For every pair of elements c_0, c'_0 of \mathbb{C}_0 , we can consider the $(n-1)$ -truncated globular set $\mathcal{C}(c_0, c'_0)$, where

$$[\mathcal{C}(c_0, c'_0)]_0 = \{c_1 \in \mathcal{C}_1 \text{ s.t. } s(c_1) = c_0, t(c_1) = c'_0\}$$

and inductively

$$[\mathcal{C}(c_0, c'_0)]_i = \{c_{i+1} \in \mathcal{C}_{i+1} \text{ s.t. } s(c_{i+1}) \in [\mathcal{C}(c_0, c'_0)]_{i-1}, t(c_{i+1}) \in [\mathcal{C}(c_0, c'_0)]_{i-1}\}$$

k -Sources, k -targets, k -starting identities and k -compositions maps, with $k = 1, \dots, n$, restrict properly to $\mathcal{C}(c_0, c'_0)$, hence it is a globular $(n-1)$ -category. By induction hypothesis hence a $(n-1)$ -category $\mathbb{C}_1(c_0, c'_0)$.

Moreover 0-composition defines $(n-1)$ -functors

$$\circ^0 : \mathcal{C}(c_0, c'_0) \times \mathcal{C}(c'_0, c''_0) \rightarrow \mathcal{C}(c_0, c''_0)$$

for every triple c_0, c'_0, c''_0 , and 0-starting identities define $(n-1)$ -functors

$$u^0(c_0) : \mathbb{I} \rightarrow \mathcal{C}(c_0, c_0)$$

These data form indeed a n -category. □

A.2 The groupoid condition

Our notion of n -groupoid corresponds, modulo the conversions recalled above, to the notion of n -groupoid of Kapranov and Voevodsky in [KV91]. According to their definition a n -groupoid is a globular strict- n -category which satisfies a so-called *groupoid-condition*. This basically says that every equation of the kind $cx = c'$ or $yc = c'$ is (weakly) solvable, when the equation makes sense. For sake of completeness this condition is recalled below.

Definition A.2 (Kapranov and Voevodsky, *Definition 1.1* [KV91]). *A n -category \mathcal{C} is called a n -groupoid if for all $i < k \leq n$ the following conditions hold*

(GR'_{i,k}, i < k - 1) For each $a \in \mathcal{C}_{i+1}$, $b \in \mathcal{C}_k$ $u, v \in \mathcal{C}_{k-1}$ such that

$$s_i(a) = t_i(u) = t_i(v), \quad u \star^i a = s_{k-1}(b), \quad v \star^i a = t_{k-1}(b)$$

there exist $x \in \mathcal{C}_k$, $\phi \in \mathcal{C}_{k+1}$ such that

$$s_k(\phi) = x \star^i a, \quad t_k(\phi) = b, \quad s_{k-1}(x) = u, \quad s_{k-1}(x) = v.$$

(GR'_{k-1,k}) For each $a \in \mathcal{C}_k$, $b \in \mathcal{C}_k$ such that

$$t_{k-1}(a) = t_{k-1}(b)$$

there exist $x \in \mathcal{C}_k$, $\phi \in \mathcal{C}_{k+1}$ such that

$$s_k(\phi) = x \star^{k-1} a, \quad t_k(\phi) = b.$$

(GR''_{i,k}, i < k - 1) For each $a \in \mathcal{C}_{i+1}$, $b \in \mathcal{C}_k$ $u, v \in \mathcal{C}_{k-1}$ such that

$$s_i(a) = t_i(u) = t_i(v), \quad a \star^i u = s_{k-1}(b), \quad a \star^i v = t_{k-1}(b)$$

there exist $x \in \mathcal{C}_k$, $\phi \in \mathcal{C}_{k+1}$ such that

$$s_k(\phi) = a \star^i x, \quad t_k(\phi) = b, \quad s_{k-1}(x) = u, \quad s_{k-1}(x) = v.$$

(GR''_{k-1,k}) For each $a \in \mathcal{C}_k$, $b \in \mathcal{C}_k$ such that

$$s_{k-1}(a) = s_{k-1}(b)$$

there exist $x \in \mathcal{C}_k$, $\phi \in \mathcal{C}_{k+1}$ such that

$$s_k(\phi) = a \star^{k-1} x, \quad t_k(\phi) = b.$$

The notion recalled above is a generalization of the definition by Street in [Str87] which included only axioms for inverses $(\text{GR}'_{k-1,k})$ and $(\text{GR}''_{k-1,k})$. Moreover it is a wider generalization of the definition by Brown and Higgins in [BH81] in which such axioms were taken in a strict form ($\phi = id$).

Notice that Kapranov and Voevodsky motivate the new axioms for they would ensure not only the existence of (weak) inverses. In fact they claim (but not prove) that all four axioms together imply the existence of a coherent system of such.

That our definition is equivalent to A.2 above is a corollary to Simpson accurate analysis developed in [Sim98], where he uses the Tamsamani's approach for the treatment of a groupoid condition for weak n -categories [Tam96].

This is resumed in the following inductive theorem-definition

Theorem A.3 (Simpson, *Theorem 2.1* [Sim98]). *Fix $n < \infty$.*

I. Groupoids *Suppose \mathcal{C} is a [globular] strict n -category. The following three conditions are equivalent (and in this case we say that \mathcal{C} is a strict n -groupoid).*

- (1) *\mathcal{C} is a n groupoid in the sense of Kapranov and Voevodsky (Definition A.2);*
- (2) *for all $x, y \in \mathcal{C}_0$, $\mathcal{C}(x, y)$ is a strict $(n - 1)$ -groupoid, and for any 1-cell $f : x \rightarrow y$ in \mathcal{C} , the two [families of] morphisms of [left and right] compositions with f are equivalences of strict $(n - 1)$ -groupoids;*
- (3) *for all $x, y \in \mathcal{C}_0$, $\mathcal{C}(x, y)$ is a strict $(n - 1)$ -groupoid, and $\tau_{\leq 1}\mathcal{C}$ is a groupoid.*

II. Truncation *If \mathcal{C} is a strict n -groupoid, then define $\tau_{\leq k}\mathcal{C}$ to be the strict k -category whose i -cells are those of \mathcal{C} for $i < k$, and whose k -cells are the equivalence classes of k -cells of \mathcal{C} under the equivalence relation that two are equivalent if there is a $(k + 1)$ -cell joining them. The fact that this is an equivalence relation is a statement about $(n - k)$ -groupoids. The set $\tau_{\leq 0}\mathcal{C}$ will also be denoted $\pi_0\mathcal{C}$. The truncation is again a k -groupoid and for n -groupoids \mathcal{C} the truncation coincide with the operation defined in [KV91].*

III. Equivalence *A morphism $F : \mathcal{C} \rightarrow \mathcal{D}$ of strict n -groupoids is said to be an equivalence if the following equivalent conditions are satisfied:*

- (a) *(this is the definition in [KV91]) F induces an isomorphism $\pi_0\mathcal{C} \rightarrow \pi_0\mathcal{D}$, and for every object $c \in \mathcal{C}$ F induces isomorphisms $\pi_i(\mathcal{C}, c) \rightarrow \pi_i(\mathcal{D}, F(c))$, where these homotopy groups are defined in [KV91];*
- (b) *F induces a surjection $\pi_0\mathcal{C} \rightarrow \pi_0\mathcal{D}$ and for every pair of objects $x, y \in \mathcal{C}$ F induces an equivalence of $(n - 1)$ -groupoids $\mathcal{C}(x, y) \rightarrow \mathcal{D}(F(x), F(y))$;*
- (c) *if u, v are i -cells in \mathcal{C} sharing source and target, and if $r : F(u) \dashrightarrow^{i+1} F(v)$ is a $(i + 1)$ -cell in \mathcal{D} , there exists an $(i + 1)$ -cell $t : u \dashrightarrow^{i+1} v$ of \mathcal{C} and a $(i + 2)$ -cell $F(t) \dashrightarrow^{i+2} r$ in \mathcal{D} (this includes the limiting cases $i = -1$ where u and v are not specified, and $i = n - 1, n$ where “ $(n + 1)$ -cell” means equality and “ $(n + 2)$ -cells” are not specified).*

Finally we can compare these conditions with our groupoid condition.

We have already discussed about the correspondence between globular and enriched versions of n -category.

Hence we can focus on characterizing conditions.

Formally our *Definition 4.11* amounts precisely to condition **I.**(2) above. What is to be checked is then the notion of equivalence. This is done by *Proposition 4.8* that is the inductive version of **III.**(c).

The notion of truncation is not directly involved, nevertheless it can be recovered by successive applications of the sesqui-functor π_0 .

A.3 System of adjoint inverses

In order to be more precise about the choices of inverses, only for this section, we give a sharper definition of equivalence that take into account directions of cells.

Definition A.4. *Let n -category morphism $F : \mathbb{C} \rightarrow \mathbb{D}$ be given.*

- *F is called equivalence of n -categories if it satisfies the following properties:*

$$\boxed{n = 0}$$

F is an isomorphism.

$$\boxed{n > 0}$$

1. *for every object d_0 of \mathbb{D} , there exists an object c_0 of \mathbb{C} and a 1-cell $d_1 : d_0 \rightarrow Fc_0$ such that for every d'_0 in \mathbb{C} , the morphism*

$$d_1 \circ - : \mathbb{D}_1(d_0, d'_0) \rightarrow \mathbb{D}_1(Fc_0, d'_0)$$

is an equivalence of $(n - 1)$ -categories, and the morphism

$$- \circ d_1 : \mathbb{D}_1(d'_0, Fc_0) \rightarrow \mathbb{D}_1(d'_0, d_0)$$

is a co-equivalence of $(n - 1)$ -categories.

2. *for every pair c_0, c'_0 in \mathbb{C} ,*

$$F_1^{c_0, c'_0} : \mathbb{C}_1(c_0, c'_0) \rightarrow \mathbb{D}_1(Fc_0, Fc'_0)$$

is an equivalence of $(n - 1)$ -categories.

- *F is called co-equivalence of n -categories if it satisfies the following properties:*

$$\boxed{n = 0}$$

F is an isomorphism.

$$\boxed{n > 0}$$

1. for every object d_0 of \mathbb{D} , there exists an object c_0 of \mathbb{C} and a 1-cell $d_1 : Fc_0 \rightarrow d_0$ such that for every d'_0 in \mathbb{C} , the morphism

$$d_1 \circ - : \mathbb{D}_1(d_0, d'_0) \rightarrow \mathbb{D}_1(Fc_0, d'_0)$$

is a co-equivalence of $(n-1)$ -categories, and the morphism

$$- \circ d_1 : \mathbb{D}_1(d'_0, Fc_0) \rightarrow \mathbb{D}_1(d'_0, d_0)$$

is an equivalence of $(n-1)$ -categories.

2. for every pair c_0, c'_0 in \mathbb{C} ,

$$F_1^{c_0, c'_0} : \mathbb{C}_1(c_0, c'_0) \rightarrow \mathbb{D}_1(Fc_0, Fc'_0)$$

is a co-equivalence of $(n-1)$ -categories.

This suggests to improve *Definition 4.2*:

Definition A.5. A 1-cell $c_1 : c_0 \rightarrow c'_0$ of a n -category \mathbb{C} is said to be weakly invertible, or simply an equivalence, if, for every object \bar{c}_0 of \mathbb{C} , the morphism

$$c_1 \circ - : \mathbb{C}_1(c'_0, \bar{c}_0) \rightarrow \mathbb{C}_1(c_0, \bar{c}_0)$$

is an equivalence of $(n-1)$ -categories, and the morphism

$$- \circ c_1 : \mathbb{C}_1(\bar{c}_0, c_0) \rightarrow \mathbb{C}_1(\bar{c}_0, c'_0)$$

is a co-equivalence of $(n-1)$ -categories.

The dual definition for a weakly co-invertible 1-cell.

When a 1-cell is weakly invertible, then it has indeed left and right weak-inverses. In fact for $c_1 : c_0 \rightarrow c'_0$,

$$c_1 \circ - : \mathbb{C}_1(c'_0, c_0) \rightarrow \mathbb{C}_1(c_0, c_0)$$

being an equivalence implies that for the 1-cell $1_{c_0} : c_0 \rightarrow c_0$ there exists a pair

$$(c_1^*, i_2 : 1_{c_1} \xrightarrow{\sim} c_1 \circ c_1^*),$$

similarly for

$$- \circ c_1 : \mathbb{C}_1(c'_0, c_0) \rightarrow \mathbb{C}_1(c'_0, c'_0)$$

being a co-equivalence implies there exists a pair

$$(c_1^\dagger, e_2 : c_1^\dagger \circ c_1 \xrightarrow{\sim} 1_{c'_1}).$$

Left and right inverses are indeed equivalent: following a classical group-theoretical argument, the 1-composition

$$c_1^\dagger = c_1^\dagger \circ 1_{c_1} \xrightarrow{c_1^\dagger \circ i_2} c_1^\dagger \circ 1_{c_1} \circ c_1^* \xrightarrow{e_2 \circ c_1^*} 1_{c_1'} \circ c_1^* = c_1^*$$

witnesses the equivalence.

Let us suppose we have chosen a system of inverse in a n -groupoid \mathbb{C} , *i.e.* for every k -cell c_k we can exhibit a weak inverse c_k^* and equivalences

$$i_{c_k} : 1 \Longrightarrow c_k \circ^{k-1} c_k^*, \quad e_{c_k} : c_k^* \circ^{k-1} c_k \Longrightarrow 1$$

Then it is always possible to get a system of adjoint inverses, according to the following

Definition/Proposition A.6. *By induction over n .*

In $0\mathbf{Gpd}$ and in $1\mathbf{Gpd}$ inverses are unique.

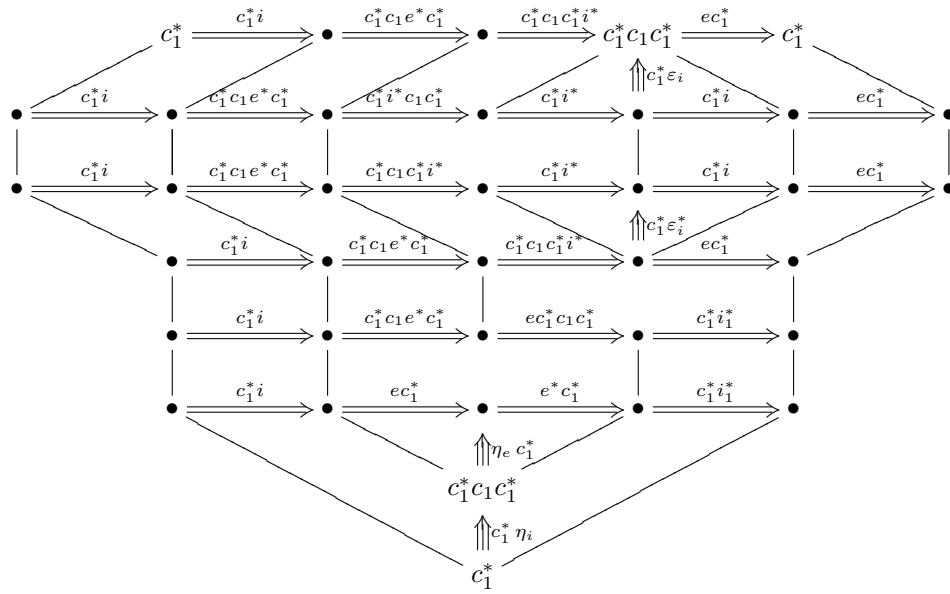
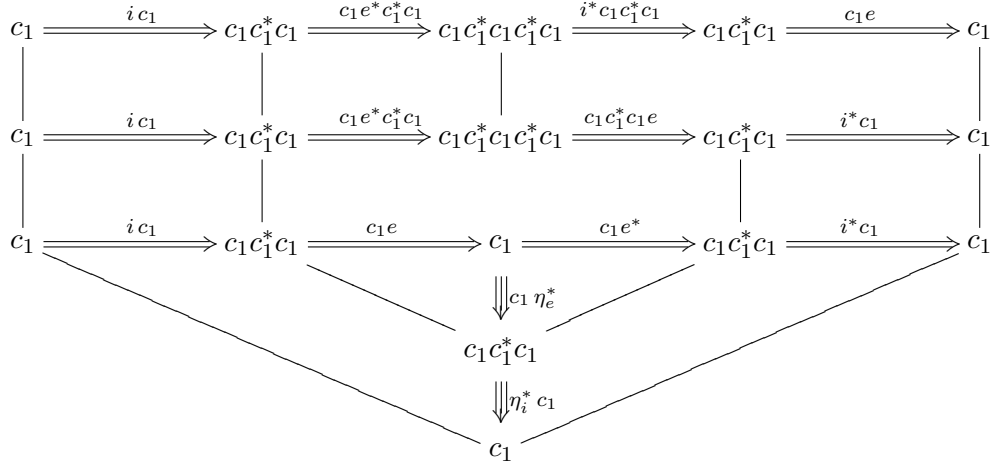
So let us suppose $n > 1$. Let us suppose further we have chosen systems of adjoint inverses in the $\text{hom-}(n-1)$ -groupoids. Then the following procedure gives a system of adjoint inverses in \mathbb{C} .

For every four-tuple $(c_1, c_1^, i_{c_1}, e_{c_1})$ one defines a new four-tuple $(c_1, c_1^*, i'_{c_1}, e'_{c_1})$ where $e'_{c_1} = e_{c_1}$ and i'_{c_1} is given by the 1-composition (0-composition is juxtaposition)*

$$1_{c_0} \xrightarrow{i_{c_1}} c_1 c_1^* \xrightarrow{c_1 e_{c_1}^* c_1^*} c_1 c_1^* c_1 c_1^* \xrightarrow{i_{c_1}^* c_1 c_1^*} c_1 c_1^*$$

Then for every 1-cell $c_1 : c_0 \rightarrow c'_0$ one has the triangular equivalences:

Proof. It suffices to paste 3-cells in the following two diagrams, where hexagons commute quite trivially, rectangles are identities. Concerning triangles, induction hypothesis provide adjoint inverses for higher dimensional cells used there.

☐

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